Dualities for prioritised default bilattices I

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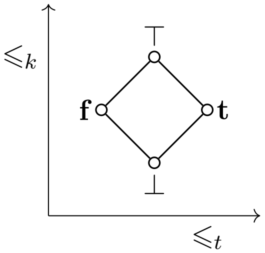
AAA100
Kraków
5–7 February 2021

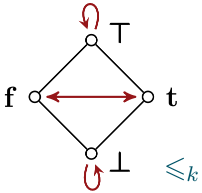
Outline

"How a computer should think"

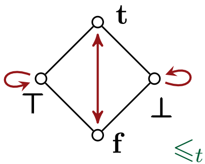
In the mid 1970's, Belnap suggested using a set of four truth values for applications in computing:



- Vertical axis represents *knowledge* order
- Horizontal axis represents *truth* order
- $\top$ = 'contradiction'
- $\bot$ = 'no information'

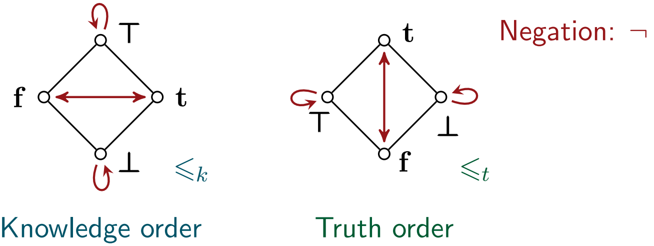


Knowledge order



Truth order

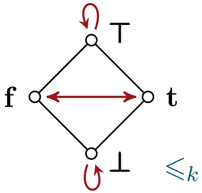
Negation: $\neg$



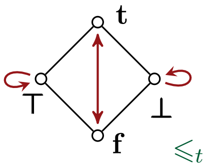
$\mathcal{FOUR} = \langle \{\mathbf{t}, \mathbf{f}, \bot, \top\}; \otimes, \oplus, \wedge, \vee, \neg \rangle$ is an example of a bilattice, that is,

- $\mathbf{L}_k := \langle \{\mathbf{t}, \mathbf{f}, \bot, \top\}; \otimes, \oplus \rangle$ is a lattice,
- $\mathbf{L}_t := \langle \{\mathbf{t}, \mathbf{f}, \bot, \top\}; \wedge, \vee \rangle$ is a lattice,
- $\neg$ preserves $\leqslant_k$ and reverses $\leqslant_t$,
- $\neg(\neg x) = x$, for all x.

Bilattices



Knowledge order



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The knowledge meet, $\otimes$, is interpreted as consensus, and the knowledge join, $\oplus$, as gullibility.

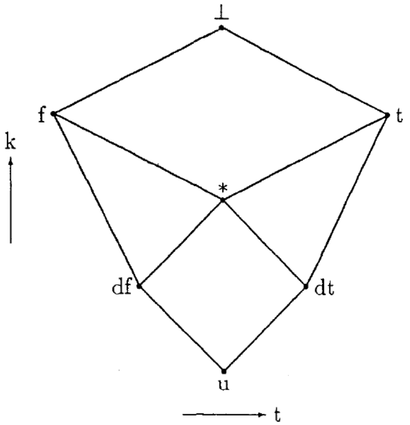

FIG. 4. D, the bilattice for default logic.

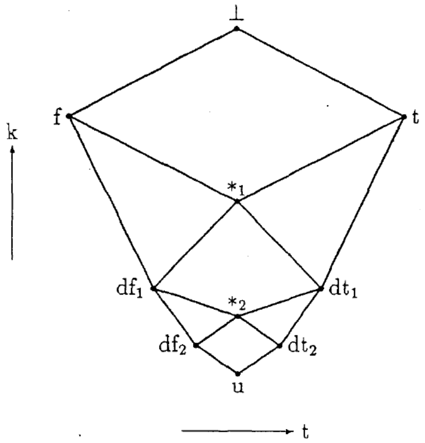

FIG. 5. The bilattice for a prioritized default logic.

Ginsberg (1988) proposed using these bilattices for default logic (left) and prioritised default logic (right).

Bilattices

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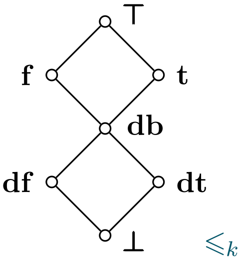

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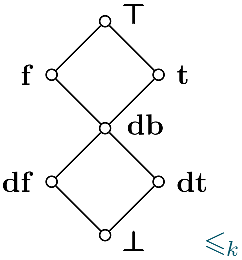

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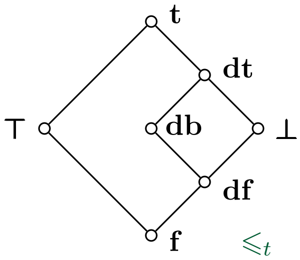

Truth order

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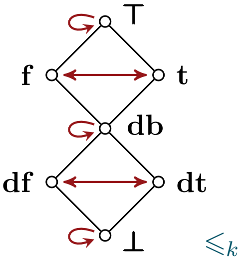

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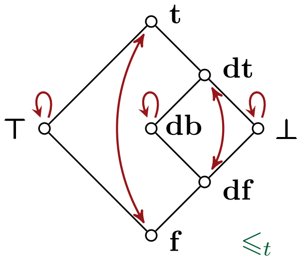
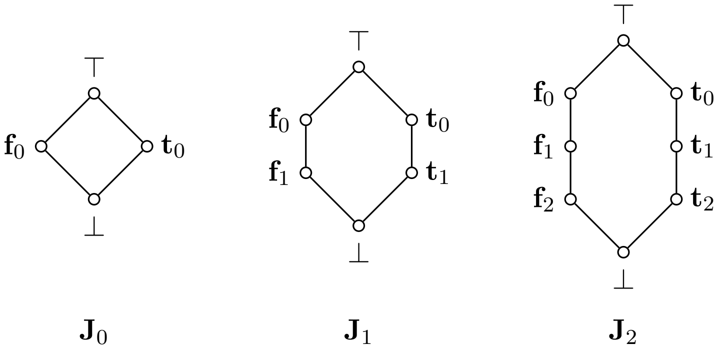


Truth order

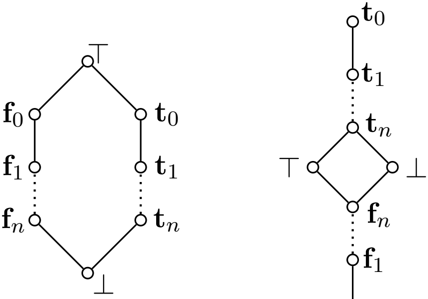
Expanding Belnap's four-element bilattice

We introduce a new family of default bilattices.



The bilattices $\mathbf{J}_0, \mathbf{J}_1$ and $\mathbf{J}_2$, drawn in their knowledge order.

The bilattice $\mathbf{J}_n$ drawn in both its knowledge (left) and truth (right) orders:



$\leqslant_k, \otimes, \oplus$ $\qquad$ $\leqslant_t, \wedge, \vee$

We let $\mathbf{J}_n = \langle J_n; \otimes, \oplus, \wedge, \vee, \neg, C\rangle$ where $C = J_n$ and $\neg\mathbf{t}_i = \mathbf{f}_i$ and $\neg\mathbf{f}_i = \mathbf{t}_i$.

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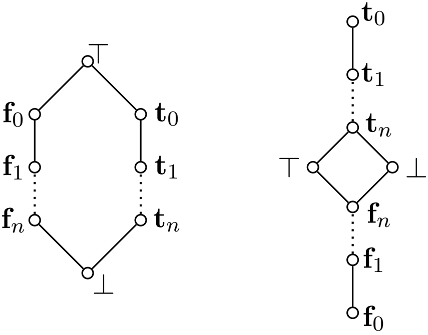

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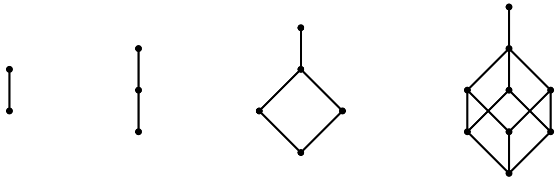
Note: $\mathbf{J}_n$ is non-distributive in its knowledge order, but distributive in its truth order.

The variety $\mathrm{Var}(\mathbf{J}_n)$

The variety Var($\mathbf{J}_n$)

Lemma

Let $n \in \omega$*. Then* $\mathrm{Con}(\mathbf{J}_n) \cong \mathbf{2}^n \oplus \mathbf{1}$ *(i.e.,* $\mathbf{2}^n$ *with a new top adjoined).*



$\mathrm{Con}(\mathbf{J}_0)$ $\mathrm{Con}(\mathbf{J}_1)$ $\mathrm{Con}(\mathbf{J}_2)$ $\mathrm{Con}(\mathbf{J}_3)$

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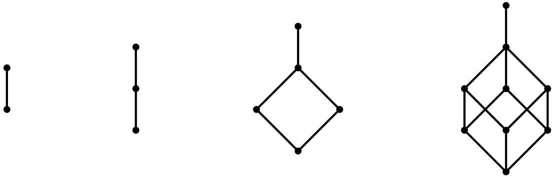

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Proposition

For $n \in \omega \setminus \{0\}$, let $\boldsymbol{\mathcal{V}}_n = \mathsf{HSP}(\mathbf{J}_n)$ be the variety generated by $\mathbf{J}_n$. Up to isomorphism, the variety $\boldsymbol{\mathcal{V}}_n = \mathsf{HSP}(\mathbf{J}_n)$ contains $n+1$ subdirectly irreducible algebras, the four-element algebra $\mathbf{M}_0$ and the six-element algebras $\mathbf{M}_k$, for $k \in \{1, \ldots, n\}$.

From the previous slide, for $n \geqslant 1$ we have:

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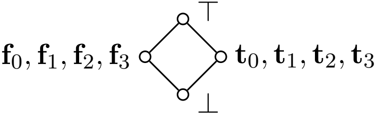
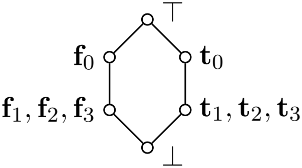
What are the $\mathbf{M}_k$?

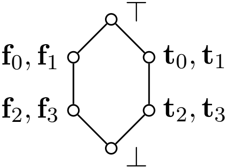
Let $n = 3$. The subdirectly irreducibles in $\boldsymbol{\mathcal{V}}_3 = \mathsf{Var}(\mathbf{J}_3)$ are:



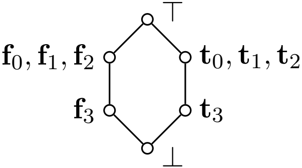
$\mathbf{M}_0$



$\mathbf{M}_1$



$\mathbf{M}_2$



$\mathbf{M}_3$

Natural duality theory

Let $\mathbf{M}$ be a finite algebra and consider $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$.

We aim to equip M with additional structure (operations, partial operations, compatible relations, the discrete topology) to obtain

$$\mathbb{M} = \langle M; \mathcal{G}, \mathcal{H}, \mathcal{R}, \mathcal{T} \rangle$$

such that $\boldsymbol{\mathcal{A}}$ is dually equivalent to $\boldsymbol{\mathcal{X}} = \mathsf{IS_c P^+}(\mathbb{M})$.

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When we are able to obtain such a structure $\mathbb{M}$, we say that $\mathbb{M}$ *yields a full duality on* $\boldsymbol{\mathcal{A}}$. We call $\mathbb{M}$ the *alter-ego* of $\mathbf{M}$.

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Example

Let $\mathbf{2}$ *be the two-element bounded distributive lattice. Then* $\mathsf{ISP}(\mathbf{2})$ *is* $\boldsymbol{\mathcal{D}}$*, the variety of bounded distributive lattices.*

The topological structure $\mathbb{2} = \langle 2; \leqslant, \mathcal{T} \rangle$ *yields a full duality on* $\boldsymbol{\mathcal{D}}$*, and* $\boldsymbol{\mathcal{P}} = \mathsf{IS_cP^+}(\mathbb{2})$ *is the class of Priestley spaces.*

We are interested in finding a class of topological structures dually equivalent to:

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To find such structures we will need to use *multi-sorted natural duality*. This means we need to equip the disjoint union $M_0 \,\dot{\cup} \cdots \dot{\cup}\, M_n$ with additional structure. More specifically, we need to find $\mathcal{G}$, $\mathcal{H}$, $\mathcal{R}$ such that

$$\mathbb{M} = \langle M_0 \,\dot{\cup} \cdots \dot{\cup}\, M_n; \mathcal{G}, \mathcal{H}, \mathcal{R}, \mathcal{T} \rangle,$$

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Fortunately for us, the theory of multi-sorted natural dualities has a very powerful result that will aid us in our task.

Theorem (**Special Multi-sorted NU Duality Theorem**)

Let $\mathbf{M}_0$*, ..., * $\mathbf{M}_n$ *be finite, pairwise non-isomorphic lattice-based algebras of the same signature, and*

$$\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\{\mathbf{M}_0, \dots, \mathbf{M}_n\}).$$

Assume that, for all $k \in \{0, \dots, n\}$*, the algebra* $\mathbf{M}_k$ *is subdirectly irreducible and every element of* $\mathbf{M}_k$ *is a constant. Define*

$$\mathbb{M} = \langle M_0 \,\dot\cup \cdots \dot\cup\, M_n; \mathcal{G}, \mathcal{R}, \mathcal{T} \rangle,$$

where $\mathcal{G} = \bigcup \{ \boldsymbol{\mathcal{A}}(\mathbf{M}_j, \mathbf{M}_k) \mid j, k \in \{0, \dots, n\} \}$ *is the set of all homomorphisms between the sorts and*

$$\mathcal{R} = \bigcup \{ \mathrm{Sub}(\mathbf{M}_j \times \mathbf{M}_k) \mid j, k \in \{0, \dots, n\} \}$$

is the set of all compatible relations between the sorts.

Then $\mathbb{M}$ *yields a multi-sorted full duality on* $\mathsf{ISP}(\{\mathbf{M}_0, \dots, \mathbf{M}_n\})$.

Fix $n \geqslant 1$ and consider $\mathcal{G} = \bigcup\{\, \boldsymbol{\mathcal{A}}(\mathbf{M}_j, \mathbf{M}_k) \mid j, k \in \{0, \ldots, n\} \,\}$ (the set of all homomorphisms between the sorts).

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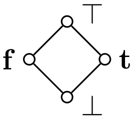

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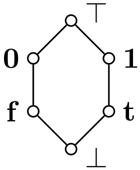

$\mathbf{M}_k$

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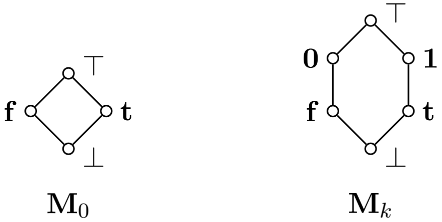

Due to the constants, the only homomorphisms between the sorts are the identity homomorphisms id_{M_k} and for $1 \leqslant k \leqslant n$, the maps $g_k : \mathbf{M}_k \to \mathbf{M}_0$ defined by $g_k(\mathbf{f}) = \mathbf{f} = g_k(\mathbf{0})$, $g_k(\mathbf{t}) = \mathbf{t} = g_k(\mathbf{1})$, $g_k(\top) = \top$, and $g_k(\bot) = \bot$.

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If $\mathcal{R} \cup \{S\}$ is a set of compatible relations, we will say that S is entailed by $\mathcal{R}$ ($\mathcal{R} \vdash S$) if whenever $\mathbb{M} = \langle M; \mathcal{R}, \mathcal{T} \rangle$ yields a duality on $\boldsymbol{\mathcal{A}}$, then $\mathbb{M}' = \langle M; \mathcal{R}\backslash\{S\}, \mathcal{T} \rangle$ also yields a duality on $\boldsymbol{\mathcal{A}}$.

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Lemma

If R_1 and R_2 are two compatible relations with $R_1 \cap R_2 \neq \emptyset$, then $R_1 \cap R_2$ is entailed by $\{R_1, R_2\}$.

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Lemma

The structure $(\operatorname{Sub}(\mathbf{M}_j \times \mathbf{M}_k), \subseteq)$ is a finite lattice, hence it is meet-generated by its meet-irreducible elements.

Let $\mathcal{F}$ be a topped intersection structure on a non-empty set X, i.e. $\mathcal{F} \subseteq \wp(X)$, $X \in \mathcal{F}$, and $\mathcal{F}$ is closed under intersections of non-empty families.

Let $x \in X$. An element Y of $\mathcal{F}$ is a value at x if Y is maximal in $\mathcal{F}$ with respect to not containing x.

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The following lemma will help us to identify the meet-irreducible elements of the lattice $\mathrm{Sub}(\mathbf{M}_j \times \mathbf{M}_k)$.

Lemma

Let $\mathcal{F}$ be a topped intersection structure on a non-empty set X. An element Y of $\mathcal{F}$ is completely meet-irreducible in the lattice $\mathcal{F}$ if and only if Y is a value at x for some $x \in X$.

Let $n \in \omega$ and let $u\colon \mathbf{J}_n \to \mathbf{A}$ and $v\colon \mathbf{J}_n \to \mathbf{B}$ be surjective homomorphisms with $\mathbf{A}$ and $\mathbf{B}$ non-trivial.

- $F_{\mathbf{A}}$ (resp. $F_{\mathbf{B}}$) is the set of false constants in $\mathbf{A}$ (resp. $\mathbf{B}$)
- $K = \{\, (u(c), v(c)) \mid c \in J_n \,\} \subseteq A \times B$
- $S_{\leqslant}$, $S_{\geqslant}$, S_{ab} are elements of $\mathrm{Sub}(\mathbf{A} \times \mathbf{B})$
- $\mathrm{Val}(a, b)$ is the set of values at (a, b)

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Theorem

The meet-irreducible elements in the lattice $\mathrm{Sub}(\mathbf{A} \times \mathbf{B})$ *are the sets* $S_{\leqslant}$ *and* $S_{\geqslant}$*, and* S_{ab}*, for all pairs* $(a, b) \in \big(F_{\mathbf{A}} \times F_{\mathbf{B}}\big)\backslash K$*.*
Indeed,

(a) $\mathrm{Val}(a, b) = \{S_{\leqslant}, S_{\geqslant}\}$, *for all* $(a, b) \in \big(A \times B\big)\backslash\big(S_{\leqslant} \cup S_{\geqslant}\big)$,

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(e) $\mathrm{Val}(a, b) = \varnothing$, *for all* $(a, b) \in K$.

The Special Multi-sorted NU Duality Theorem required

$$\mathcal{R} = \bigcup \{ \mathrm{Sub}(\mathbf{M}_j \times \mathbf{M}_k) \mid j, k \in \{0, \dots, n\} \}$$

(the set of all compatible relations between the sorts M_0, ... M_n) to be included in the structure on $\mathbb{M}$.

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Lemma

If A, B, C and D are sorts, $g\colon A \to C$, $h\colon B \to D$ and $S \subseteq C \times D$, then define

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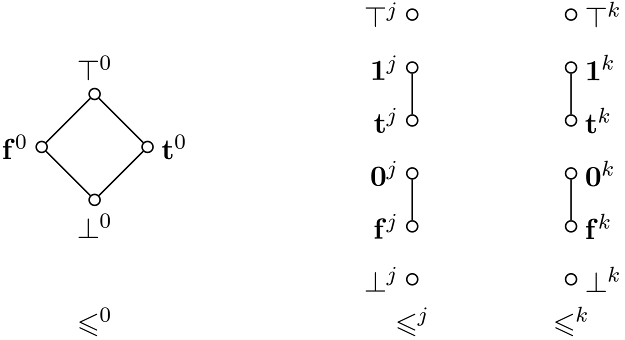
We use this lemma with the maps g_k $(1 \leqslant k \leqslant n)$ to remove some meet-irreducible elements of $\operatorname{Sub}(\mathbf{M}_j \times \mathbf{M}_k)$ from the alter ego.

Final structure of the alter-ego $\mathbb{M}_n$

Let $n \in \omega \backslash \{0\}$ and let $j, k \in \{1, \ldots, n\}$ with $j < k$.

The set $M_0 \,\dot\cup \cdots \dot\cup\, M_n$ will have the following structure:



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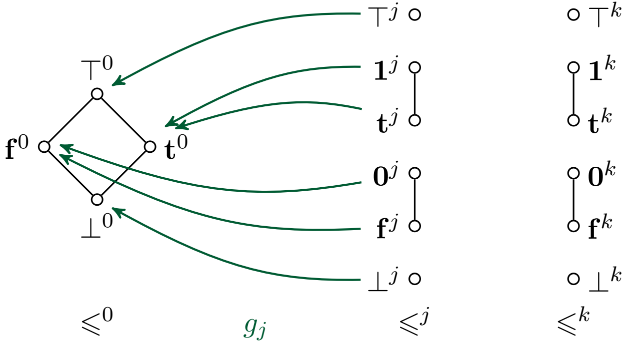

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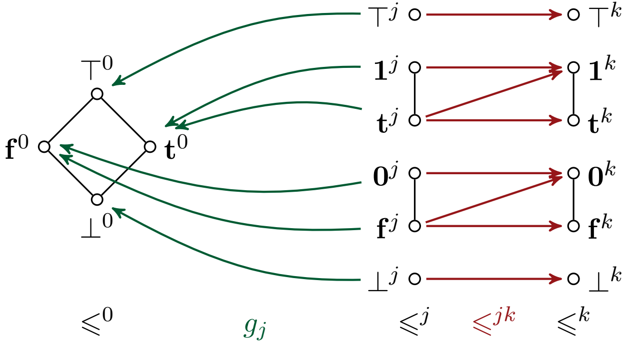


For all $k \in \{1, \ldots, n\}$, let $g_k \colon \mathbf{M}_k \to \mathbf{M}_0$ be the homomorphism that maps $\mathbf{f}$ and $\mathbf{0}$ to $\mathbf{f}$ and maps $\mathbf{t}$ and $\mathbf{1}$ to $\mathbf{t}$.

Theorem

Let $n \in \omega \backslash \{0\}$. Define the multi-sorted alter ego

$$\mathbb{M}_n = \langle M_0 \,\dot{\cup}\, M_1 \,\dot{\cup} \cdots \dot{\cup}\, M_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T} \rangle,$$

where

$$\mathcal{G}_{(n)} = \{\, g_k \mid k \in \{1, \ldots, n\} \,\}, \text{ and}$$
$$\mathcal{S}_{(n)} = \{\, \leqslant^k \mid k \in \{0, \ldots, n\} \,\} \cup \{\, \leqslant^{jk} \mid j, k \in \{1, \ldots, n\} \text{ with } j < k \,\}.$$

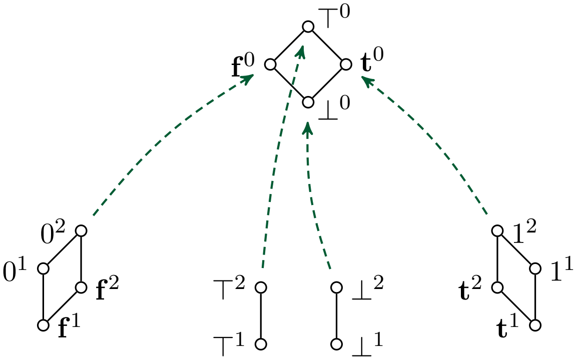
1. *The alter ego $\mathbb{M}_n$ yields a full (optimal) duality on $\boldsymbol{\mathcal{V}}_n = \mathsf{Var}(\mathbf{J}_n) = \mathsf{ISP}(\{\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_n\})$.*
2. *The category $\mathsf{IS_cP^+}(\mathbb{M}_n)$ is dually equivalent to $\boldsymbol{\mathcal{V}}_n$.*
3. $|\mathcal{S}_{(n)} \cup \mathcal{G}_{(n)}| = \frac{1}{2}(n^2 + 3n + 2)$.

Example

The multi-sorted alter ego

$$\mathbb{M}_2 = \langle M_0 \,\dot\cup\, M_1 \,\dot\cup\, M_2; g_1, g_2, \leqslant^0, \leqslant^1, \leqslant^2, \leqslant^{12}, \mathcal{T}\rangle$$

yields a full duality on the variety $\boldsymbol{\mathcal{V}}_2 = \mathsf{Var}(\mathbf{J}_2)$.



Thank you