Generalized quadratic functional equations on ternary quasigroups

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Quasigroups

Definition (Quasigroups)

Groupoid $(Q; \cdot)$ is **quasigroup**
iff for all $a, b \in Q$, there are unique $x, y \in Q$
such that $x \cdot a = b$ and $a \cdot y = b$.

Ternary groupoid $(Q; F)$ is **ternary quasigroup**
(**3–quasigroup** for short)
iff for all $a, b, c \in Q$,
there are unique $x, y, z \in Q$
such that $F(x, a, b) = c$, $F(a, y, b) = c$ and $F(a, b, z) = c$.

Quasigroups

Definition (Loops)

Loop is a quasigroup with **unit** e:

$$e \cdot x = x \cdot e = x.$$

3–loop is a 3–quasigroup with **unit** e:

$$F(e, e, x) = F(e, x, e) = F(x, e, e) = x.$$

Theorem

Associative quasigroup is a group.

Associative and commutative quasigroup is an Abelian group.

Quasigroups

Definition (Parastrophy)

Let $(Q;\cdot)$ be a quasigroup.
Operation $\cdot$ and five operations $\backslash, /, *, \backslash\backslash, //$ defined by:

$$x\backslash z = y \quad \Leftrightarrow \quad x\cdot y = z \quad \Leftrightarrow \quad z/y = x$$

$$z\backslash\backslash x = y \quad \Leftrightarrow \quad y * x = z \quad \Leftrightarrow \quad y//z = x$$

are **parastrophes** of $\cdot$ (and of each other).

Let $(Q;F)$ be a 3–quasigroup.
24 operations $F^\pi(\pi \in S_4)$ defined by:

$$F^\pi(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) = x_{\pi(4)} \quad \Leftrightarrow \quad F(x_1, x_2, x_3) = x_4$$

are **parastrophes** of F.

Quasigroups

In groups:

$$x * y = y \cdot x$$

$$x \backslash y = x^{-1} \cdot y \qquad x / y = x \cdot y^{-1}$$

$$x \backslash\!\backslash y = y^{-1} \cdot x \qquad x /\!/ y = y \cdot x^{-1}$$

Quasigroups

Definition (Isotopy)

Quasigroups $(Q;\cdot)$ and $(Q';\circ)$ are **isotopic**
(also: quasigroup operations $\cdot$ and $\circ$ are **isotopic**)
if there are bijections $\alpha,\lambda,\varrho : Q \longrightarrow Q'$ such that

$$\alpha(x\cdot y) = \lambda x \circ \varrho y.$$

3–quasigroups $(Q;F)$ and $(Q';G)$ are **isotopic**
(also: 3–quasigroup operations F and G are **isotopic**)
if there are bijections $\alpha,\lambda,\mu,\varrho : Q \longrightarrow Q'$ such that

$$\alpha F(x,y,z) = G(\lambda x,\mu y,\varrho z).$$

Quasigroups

Theorem

Isotopy is an equivalence relation among (3–)quasigroups.

Theorem

Isotopic groups are isomorphic.

Theorem

Every (3–)quasigroup is isotopic to a (3–)loop.

Quasigroups

Isostrophy = isotopy + parastrophy
and/or
parastrophy + isotopy

Equations

We consider equations which are:

- functional
- on (binary and) ternary quasigroups
- generalized
- quadratic

Generalized quadratic equations

Quadratic binary equations are introduced in:

A. Krapež:
Strictly quadratic functional equations on quasigroups I,
Publ. Inst, Math. (Beograd) (N.S.) 29 (43), (1981), 125–138.

Generalized quadratic equations

The equivalence relation $\sim$
related to isostrophy of quasigroup operations in $s = t$ is defined.

The case $\sim = \Box$ (all operations are mutually isostrophic) is solved.

Generalized quadratic equations

The general case for binary quasigroups
is solved by S. Krstić in his PhD Thesis:

Quadratic quasigroup identities (Serbian),
University of Belgrade (1985).

This is an atempt to generalize his results to ternary case.

Krstić graphs

To solve quadratic equation $s = t$,
S. Krstić used connected cubic *multigraph* $K(s = t)$.

The vertices of $K(s = t)$ are operation symbols from $s = t$.

The edges of $K(s = t)$ are subterms of s and t,
including s and t which are considered to be a single edge.

Likewise, any variable which appears twice in $s = t$
is taken to be a single edge.

If $F(\sigma, \tau)$ is a subterm of s or t,
then the vertex F is incident to edges $\sigma, \tau, F(\sigma, \tau)$ and no other.

KL–graphs

We define **Krstić like multigraph** (KL–graph) $K(s = t)$ to be just like Krstić graph, allowing ternary operations in $s = t$ and consequently vertices of degree 4 in $K(s = t)$:

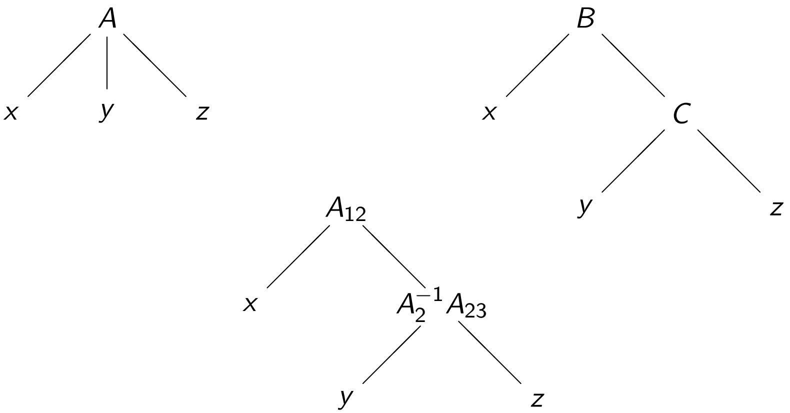
If $F(\sigma, \tau, \omega)$ is a subterm of s or t, then the vertex F is incident to edges $\sigma, \tau, \omega, F(\sigma, \tau, \omega)$ and no other.

Quadratic equations and KL–graphs

Theorem (Parastrophic equivalence)

Generalized quadratic quasigroup functional equations Eq and Eq' are parastrophically equivalent
iff
graphs $K(Eq)$ and $K(Eq')$ are isomorphic.

Reducibility



Step 1

If equation $s = t$ is reducible
then reduce one reducible operation
repeat
until obtained equation is irreducible.

Step 1

After Step 1,
we have a system of several identities such as:

$$A(x, y, z) = B(C(u, v), w) \quad (\{x, y, z\} = \{u, v, w\})$$

and a single irreducible equation $s' = t'$. The system is equivalent to $s = t$.

Step 2

If equation $s = t$ is irreducible and
if $\sim \neq \square$
then split equation
repeat
until $\sim_i = \square_i$ in all obtained equations $s_i = t_i$.

After Step 2:

$$s = t \quad \Leftrightarrow \quad \bigwedge_i s_i = t_i .$$

Step 3

Theorem

If equation $s = t$ has a single $\sim$–class
then
all operations from $s = t$ are of the same arity.

Theorem

If equation $s = t$ has a single $\sim$–class
then
either all oprations in $s = t$ are binary
or there are just one or two operations in $s = t$.

Step 3

If equation $s = t$ has a single $\sim$–class, then:

- All operations from $s = t$
  are isostrophic to the same (3–)loop L.
- L is a group
  iff there are more than two operations in $s = t$
  iff tetrahedron K_4 is embeddable in $K(s = t)$.
- L is an Abelian group
  iff $K(s = t)$ is not planar
  iff the complete bipartite graph $K_{3,3}$
  is embeddable in $K(s = t)$.

Equations with 3–quasigroups

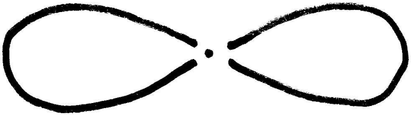
Equations under investigation are considered in:

F. Sokhatsky, H. Krainichuk, A. Tarasevych:
A classification of generalized functional equations on ternary quasigroups,
Visnyk DonNU, A: Natural Sciences no. 1-2, 2017.

F. M. Sokhatsky, A. Tarasevych:
Classification of generalized ternary quadratic quasigroup functional equations of the length three,
Carpathian Math. Publ. 11(1), 2019, 179–192.
http://www.journals.pu.if.ua/index.php/cmp

Quadratic equations and KL–graphs, $n = 2$

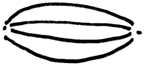
$$A(x, x, y) = y$$



Quadratic equations and KL–graphs, $n = 3$

$$A(x, x, y) = E(y, z, z)$$

$$A(x, y, z) = E(x, y, z)$$

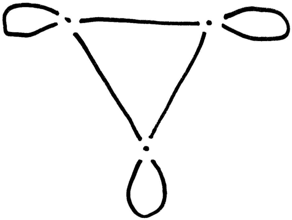


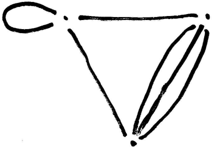
Quadratic equations and KL–graphs, $n = 4$
Irreducible equation

$$A(x, B(x, y, y), z) = E(z, u, u)$$

Quadratic equations and KL–graphs, $n = 4$
Irreducible equations

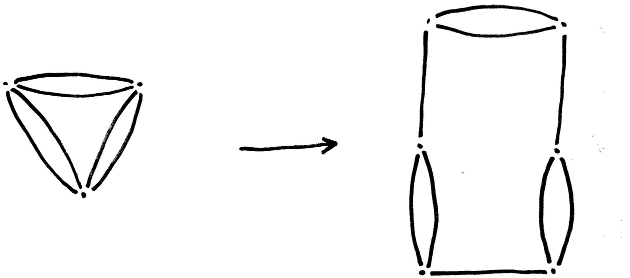
$$A(B(x,y,y),z,z) = E(x,u,u) \qquad A(B(x,y,z),u,u) = E(x,y,z)$$





Quadratic equations and KL–graphs, $n = 4$
Reducible equation

$$A(x, B(x, y, z), u) = E(y, z, u)$$



Quadratic equations and KL–graphs

Equation:

$$A(x,x,y)=y \tag{1}$$

has two obvious solutions:

- If $A(x,x,y)=B(C(x,x),y)$ then $C(x,x)=e$, $B^{(13)}(y,y)=e$ is a solution of (1).
- If $A(x,x,y)=B(x,C(x,y))$ then there is a loop $\cdot$ such that C is isotopic to $\cdot$ while B is isotopic to $\backslash$.

Problem

Is there a solution of (1) such that A is irreducible?