Derivation on Hyperlattices revisited

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Motivation

The concept of derivation was introduced by Posner on ring theory in 1957 as an additive map that satisfy the Leibniz's formula.

K. Kaya (1987) and H.E. Bell et al. (1989) have studied derivations in rings and prime rings.

Szasz (1975) have introduced and developed the theory of derivations in lattice structure.

L. Ferrari (2001) extended these concepts to lattices and he embedded any lattice having some additional properties into the lattice of its derivations.

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- In 2018, Ozbal et al. [6] defined a derivation on hypertreillis as being for a hypertreillis L a map d from L to L which only satisfies the first condition.
- We keep here the definition of Ozbal[6] and we introduce the notion of joinitive derivation or sup-derivation, then we establish some properties of these notions as well as the relations between them.

Hyperlattice, Konstantinidou and Mittas, 1977

Let L be a non-empty set with $\vee : L \times L \longrightarrow P(L)\backslash\{\emptyset\}$ a hyperoperation and $\wedge : L \times L \longrightarrow L$ an operation.
Then $\mathcal{L} = (L, \vee, \wedge)$ is a **hyperlattice** if the following conditions are satisfy. For all $a, b, c \in L$

(hl1) $a \wedge b = b \wedge a \,;\, a \vee b = b \vee a$.

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(hl2) $a \in a \vee a$; $a = a \wedge a$.

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(hl2) $a \in a \vee a \,;\, a = a \wedge a$.

(hl3) $a \in [a \vee (a \wedge b)] \cap [a \wedge (a \vee b)]$.

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(hl4) $(a \wedge b) \wedge c = a \wedge (b \wedge c) \,;\, (a \vee b) \vee c = a \vee (b \vee c)$.

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(hl4) $(a \wedge b) \wedge c = a \wedge (b \wedge c) \, ; (a \vee b) \vee c = a \vee (b \vee c)$.

(hl5) $a \in a \vee b \Rightarrow b = a \wedge b$.

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Where, for all non-empty subsets A and B of L and $a \in L$,

$$A \wedge B = \{b \wedge c, b \in A, c \in B\}, \qquad A \vee B = \bigcup\{b \vee c, b \in A, c \in B\},$$
$$a \vee B =: \{a\} \vee B, \qquad a \wedge B =: \{a\} \wedge B.$$

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Let $\mathcal{L} = (L, \vee, \wedge)$ be a **hyperlattice**,

- condition $(hl5)$ is an equivalence and it induces an **order relation** $\leqslant$ on $\mathcal{L}$ define by $a \leqslant b \Leftrightarrow a = a \wedge b \Leftrightarrow b \in a \vee b$.

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- an element $b \in L$ is called a **complement** of $a \in L$ if $a \wedge b = 0$ and $1 \in a \vee b$.
- An element $a \in L$ is called a **scalar element** if for all $x \in L$, the set $a \vee x$ has only one element.

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- $\mathcal{L}$ is called a **distributive hyperlattice** if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds for every $a, b, c \in L$.

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- $\mathcal{L}$ is called a **s-distributive** if for any $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

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- $\mathcal{L}$ is called a **s-distributive** if for any $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
- $\mathcal{L}$ is said to be 2-**torsion free**, if for any $x \in L$, $0 \in x \vee x$ implies $x = 0$.

Examples of hyperlattice

(1) Considering the set of natural numbers $\mathbb{N}$ with the meet $\wedge$ and the hyperoperation $\vee$ defined by: for all $x, y \in \mathbb{N}$, $x \wedge y := min\{x, y\}$ and $x \vee y := \{m \in \mathbb{N}, m \geqslant max\{x, y\}\}$. Then $(\mathbb{N}, \vee, \wedge)$ is a non-distributive hyperlattice, with bottom element, but no top element.

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(2) (**Nakano Hyperlattice**, see [1]) Consider a lattice $(L, \vee, \wedge)$, the Nakano hyperoperation $\sqcup$ is define on L by $x \sqcup y = \{z \in L\,;\, x \vee y = x \vee z = y \vee z\}$, for all $x, y \in L$. Therefore, $(L, \sqcup, \wedge)$ is a hyperlattice, which is distributive when $(L, \vee, \wedge)$ is distributive.

Ideals and Filters of hyperlattices

Let $\mathcal{L} = (L, \vee, \wedge)$ be a **hyperlattice**, A non-empty subset I of L is called an **ideal** of $\mathcal{L}$ if for all $a, b \in L$,

(HI1) $a, b \in I$ implies $a \vee b \subseteq I$.

(HI2) If $a \in I$ and $b \leqslant a$, then $b \in I$.

Moreover, a proper ideal I of $\mathcal{L}$ (i.e., $I \neq L$) is called a **prime ideal** of $\mathcal{L}$ if $a \wedge b \in I$ implies $a \in I$ or $b \in I$ for all $a, b \in L$.

Ideals and Filters of hyperlattices

Let $\mathcal{L} = (L, \vee, \wedge)$ be a **hyperlattice**, A nonempty subset F of L is called a **filter** of $\mathcal{L}$ if for all $a, b \in L$,

(HF1) $a, b \in F$ implies $a \wedge b \in F$.

(HF2) if $a \in F$ and $a \leqslant b$, then $b \in F$.

A proper filter F of $\mathcal{L}$ (i.e., $F \neq L$) is called a **prime filter** of $\mathcal{L}$ if $a, b \in L$ and $(a \vee b) \cap F \neq \emptyset$ implies $a \in F$ or $b \in F$.
Let $I(\mathcal{L})$ and $F(\mathcal{L})$ be respectively the set of all ideals and the set of all filters of the hyperlattice $\mathcal{L}$.

Ideals and Filters of hyperlattices

(1) When $\mathcal{L}$ is a bounded hyperlattice, with bottom element 0 and unit element 1, every ideal I of $\mathcal{L}$ contains 0 and every filter F of $\mathcal{L}$ contains 1. But $\{0\}$ is not always an ideal of $\mathcal{L}$.

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(2) If $\mathcal{L}$ is a distributive hyperlattice, then $\{0\}$ is always an ideal of $\mathcal{L}$ and for all $a \in L$, $(a] := \{x \in L \mid x \leqslant a\}$ is an ideal of $\mathcal{L}$.

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(5) Let I and F be respectively a proper ideal and a proper filter of $\mathcal{L}$. I is a prime ideal of $\mathcal{L}$ if and only if $L \backslash I$ is a prime filter of $\mathcal{L}$.

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a simple characterization of the ideal generated by a subset in distributive hyperlattices.

Proposition

Let X be a nonempty subset of a distributive hyperlattice $\mathcal{L}$. Then

$$\langle X \rangle = \{x : x \in (a_1] \vee (a_2] \vee \ldots \vee (a_n], \textit{ for some } a_1, \ldots, a_n \in X \textit{ and } n \geqslant 1\}$$

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Corollary

Let I be an ideal of a distributive hyperlattice $\mathcal{L}$ and $a \in L$, then

$$\langle I \cup \{a\} \rangle = \{x \in L : x \in \alpha \vee \beta \textit{ for some } \alpha \in I; \beta \leqslant a\} = I \vee (a]$$

Homomorphism of hyperlattices

The concept of homomorphism between hyperlattices have been expressed in terms of set-inclusion and of equalities, see [4], [5], [1].

Definition

Let $\mathcal{L}$ *and* $\mathcal{L}'$ *be two hyperlattices and* $f : L \longrightarrow L'$ *be a mapping.*

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(1) *f is said to be a **hyperlattice homomorphism** if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) \subseteq f(x) \vee f(y)$ for all $x, y \in L$.*

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(2) *f is said to be a **strong homomorphism** of a hyperlattice, if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$.*

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*If f is a bijection hyperlattice homomorphism (strong homomorphism), then f is said to be a **hyperlattice isomorphism** (**strong isomorphism**).*

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(7) ***Prime derivation****, if the $ker(d) := \{x \in L\, ;\, d(x) = 0\}$ is a prime ideal of $\mathcal{L}$.*

Examples of derivations on hyperlattices (1)

Let $\mathcal{L}$ be a hyperlattice.

(1) If there is a least element 0 in $\mathcal{L}$, then and $d : L \longrightarrow L$ a map such that $d(x) = 0$, for all $x \in L$ is a derivation on $\mathcal{L}$, called a **trivial derivation** and it is isotone, contractive, non-extensive and joinitive.

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(2) The identity function, $d(x) = x$, for every $x \in L$ is a derivation on $\mathcal{L}$ called the **identity derivation** and it is contractive, extensive, isotone and strong-joinitive.

Examples of derivations on hyperlattices (2)

Example 2:
Considering $L = \{0, a, b, c, 1\}$ with the meet $\wedge$ and the hyperoperation $\vee$. We have the Cayley tables of $\wedge$ and $\vee$ below with the following maps defined as follows:

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	{0}	{a}	{b}	{c}	{1}
a	{a}	{0,a}	{0,b}	{0,c}	{0,1}
b	{b}	{0,b}	{0,a,b}	{0,1}	{0,c,1}
c	{c}	{0,c}	{0,1}	{0,a,c}	{0,b,1}
1	{1}	{0,1}	{0,c,1}	{0,b,1}	L

Table: Table of an example of a distributive hyperlattice.

Then, $\mathcal{L}_2 = (L, \vee, \wedge)$ is a distributive bounded hyperlattice.

Examples of derivations on hyperlattices (2)

Example 2:

$\mathcal{L}_2 = (L, \vee, \wedge)$ is a distributive bounded hyperlattice. Consider the following maps:

$d_1 : L \longrightarrow L$ by $d_1(a) = a$, $d_1(b) = b$, $d_1(0) = d_1(c) = d_1(1) = 0$;

d_1 is a contractive derivation, but not isotone, not extensive, not joinitive

$d_2 : L \longrightarrow L$ by $d_2(0) = 0$, $d_2(a) = d_2(b) = d_2(c) = d_2(1) = a$;

d_2 is isotone, contractive, joinitive and strong-joinitive.

Examples of derivations on hyperlattices (3)

Example 3:
Considering the set of natural numbers $\mathbb{N}$ with the meet $\wedge$ and the hyperoperation $\vee$ defined as: for all $x, y \in \mathbb{N}$,
$x \wedge y := min\{x, y\}$ and $x \vee y := \{m \in \mathbb{N}, m \geqslant max\{x, y\}\}$. Let $a \in \mathbb{N}$ and consider the map define on $\mathbb{N}$ by $d : \mathbb{N} \longrightarrow \mathbb{N}$ such that :

$$d(x) = \begin{cases} a, & x \leqslant a \\ x, & \text{if not} \end{cases}$$

Then d is an isotone, non-contractive, extensive and strong-joinitive derivation on the non-distributive hyperlattice $(\mathbb{N}, \vee, \wedge)$.

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(6) For all $x, y \in L$, if $y \leqslant x$ and $d(x) = x$, then $d(y) = y$.

Principal Derivation on Hyperlattices

Let $\mathcal{L}$ be a hyperlattice with top element 1 and d be a derivation on $\mathcal{L}$. Then the following conditions are equivalent :

(1) d is an isotone and contractive derivation;

(2) d is principal ($d(x) = x \wedge d(1)$);

(3) $d(x \wedge y) = x \wedge d(y) = y \wedge d(x)$, for all $x, y \in L$.

Principal Derivation on Hyperlattices

Let $D(\mathcal{L}) := \{d_a | a \in L\}$ be the set of all principal derivations on a bounded hyperlattice $\mathcal{L}$.
Recall that, for all $a, b \in L$, $d_a = d_b \Leftrightarrow a = b$ and $d_a \leqslant d_b \Leftrightarrow a \leqslant b$, where $d_a \leqslant d_b$ means $d_a(x) \leqslant d_b(x)$, for all $x \in L$.

Principal Derivation on Hyperlattices

Consider on $D(\mathcal{L})$ the operation $\sqcap$ and the hyperoperation $\sqcup$ define by:

$$\begin{aligned} \sqcup : D(\mathcal{L}) \times D(\mathcal{L}) &\longrightarrow P(D(\mathcal{L})) - \varnothing \\ (d_a, d_b) &\mapsto d_a \sqcup d_b = \{d_\alpha : \alpha \in a \vee b\}. \end{aligned}$$

and

$$\begin{aligned} \sqcap : D(\mathcal{L}) \times D(\mathcal{L}) &\longrightarrow D(\mathcal{L}) \\ (d_a, d_b) &\mapsto d_a \sqcap d_b = d_{a \wedge b} \end{aligned}$$

Denote $O_{D(\mathcal{L})}$ and $1_{D(\mathcal{L})}$ respectively the trivial derivation and the identity derivation on $\mathcal{L}$.

Principal Derivation on Hyperlattices

$(D(\mathcal{L}), \sqcup, \sqcap, 0_{D(\mathcal{L})}, 1_{D(\mathcal{L})})$ is a bounded hyperlattice.
Furthermore, $(D(\mathcal{L}), \sqcup, \sqcap)$ is distributive if and only if $\mathcal{L}$ is distributive
and
$(D(\mathcal{L}), \sqcup, \sqcap, 0_{D(\mathcal{L})}, 1_{D(\mathcal{L})})$ is strong-isomorphic to $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ with the strong-isomorphism given by

$$\begin{array}{rcl} \varphi : L & \longrightarrow & D(\mathcal{L}) \\ a & \mapsto & \varphi(a) = d_a \end{array}$$

Fixed of Derivations on Hyperlattices

Let d be a derivation on a hyperlattice $\mathcal{L}$. We denote by $Fix_d(L)$ the set of all elements of L fixed by d, i.e., $Fix_d(L) := \{x \in L | d(x) = x\}$. We have the following :

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(4) Let I be a prime ideal of $\mathcal{L}$. Then, there exist a derivation d on $\mathcal{L}$ such that $Fix_d(L) = I$.

Consider for any $a \in I$, the map $d : L \longrightarrow L$ such that for all $x \in L$ $d(x) = \begin{cases} x, & x \in I \\ x \wedge a, & x \in L - I \end{cases}$

Conclusion

In this work, we have redefined the notion of derivations on hyperlattices, investigated their properties and studied the relation between isotone, joinitive and contractive derivations. We have also investigated the algebraic structure of the set of principal derivations.

we discuss the relationship between prime ideals and derivations in hyperlattice.

As futur work, we will look if its possible to characterize the distributivity by derivation as it was done by Ferrari [2]

THANK YOU FOR YOUR ATTENTION !!!!!!!!

- A. Amroune and A. Oumhani, A representation theorem of bounded distributive hyperlattice; Quasigroups and related systems, 26(2018), 1-12.
- L. Ferrari, On derivations of lattices, Pure Mathematics and Applications, 12(2001), No.45, 365- 382.
- B. B. N. Koguep, C. Lele and J. B. Nganou, Normal hyperlattices and pure ideals of hyperlattices, Asian-European Journal of Mathematics, 9(1)(2016), 1650020.
- M. Konstantinidou and J. Mittas, An introduction to the theory of hyperlattice, Mathematica balkanika, 7(1977), 187-193.
- A. Rahnamai-Barghi, The prime ideal theorem and semi-prime ideals in meet-hyperlattices, Italian Journal of Pure and Applied mathematics, 5(1999), 53-60.

S. A. Ozbal and A. Firat, On Derivations on hyperlattices, Sigma Journal of Engineering and Natural Science, 9(1)(2018), 77-84.

J. Wang, Y. Jun, X. Xin and Y. Zou, On derivation of bounded hyperlattice, Journal of Mathematical Research with Applications, 36(2016), 151-161.