Fraïssé-type theorem for polymorphism-homogeneous structures

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Joint work with M. Pech

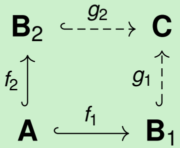
Classical tools

σ – a countable relational signature

A class of σ-structures $\mathscr{C}$ has:

- **JEP** – if for all $\mathbf{A}, \mathbf{B} \in \mathscr{C}$ there exists a $\mathbf{C} \in \mathscr{C}$: $\mathbf{A} \hookrightarrow \mathbf{C}$ and $\mathbf{B} \hookrightarrow \mathbf{C}$.

- **AP** – if for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathscr{C}$ and all $f_1 : \mathbf{A} \hookrightarrow \mathbf{B}_1$, $f_2 : \mathbf{A} \hookrightarrow \mathbf{B}_2$ there exist a $\mathbf{C} \in \mathscr{C}$, and $g_1 : \mathbf{B}_1 \hookrightarrow \mathbf{C}$ and $g_2 : \mathbf{B}_2 \hookrightarrow \mathbf{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.



Let $\mathcal{U}$ be a σ-structure.

$$\mathrm{Age}(\mathcal{U}) = \{\mathbf{A} \mid \mathbf{A} \text{ is finite } \& \ \mathbf{A} \hookrightarrow \mathcal{U}\}$$

Homogeneity

- A σ-structure is called **homogeneous** if every isomorphism between its finite substructures extends to an automorphism.

Theorem (Fraïssé 1954)

(1) $\mathcal{U}$ is countable & homogeneous $\implies \mathrm{Age}(\mathcal{U})$ has the AP.

(2) If $\mathscr{C}$ is a class of finite σ-structures, closed under isomorphisms and substructures, with countably many isomorphism types, having the JEP and AP then there exists a countable homogeneous $\mathcal{U}$ with age $\mathscr{C}$.

(3) $\mathcal{U}, \mathcal{V}$ - countable homogenous σ-structures

$$\mathrm{Age}(\mathcal{U}) = \mathrm{Age}(\mathcal{V}) \implies \mathcal{U} \cong \mathcal{V}.$$

Let's loosen things up a bit

A σ-structure $\mathcal{U}$ is called:

- **homomorphism-homogeneous** if every local homomorphism of $\mathcal{U}$ extends to an endomorphism of $\mathcal{U}$.

An n-ary **polymorphism** of $\mathcal{U}$ is any homomorphism from $\mathcal{U}^n \to \mathcal{U}$.

- **polymorphism-homogeneous** if every partial polymorphism of $\mathcal{U}$ with finite domain extends to a global polymorphism of $\mathcal{U}$.

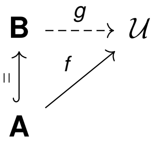
Recall (Ch. Pech, M. Pech; 2015)

$\mathcal{U}$ is polymorphism-homogeneous $\iff$ $\mathcal{U}^n$ is homomorphism-homogeneous, $\forall n > 0$.

Extension properties for HH & PH

A σ-structure $\mathcal{U}$ has:

- the **HEP** if for all $\mathbf{A} \leqslant \mathbf{B} \in \mathrm{Age}(\mathcal{U})$ and $f\colon \mathbf{A} \to \mathcal{U}$, $\exists g\colon \mathbf{B} \to \mathcal{U}$ extending f.



- the **nPEP** if for all $\mathbf{A} \leqslant \mathbf{B} \in \mathrm{Age}(\mathcal{U}^n)$ and $f\colon \mathbf{A} \to \mathcal{U}$, $\exists\, g\colon \mathbf{B} \to \mathcal{U}$ extending f.

$\mathcal{U}$ has the **PEP** $\iff$ $\mathcal{U}$ has the nPEP, $\forall n > 0$.

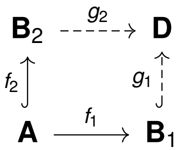
$\mathcal{U}$ has the PEP $\iff$ $\mathcal{U}^n$ has the HEP, for all $n > 0$.

$\mathcal{U}$ is polymorphism-homogeneous $\iff$ $\mathcal{U}$ has the PEP.

The key ingredient

Let $\mathscr{C}$ be a class of finite σ-structures.

$\mathscr{C}$ has the **HAP** if for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathscr{C}$, $f_1 : \mathbf{A} \to \mathbf{B}_1$ and $f_2 : \mathbf{A} \hookrightarrow \mathbf{B}_2$, there exists a $\mathbf{D} \in \mathscr{C}$, $g_1 : \mathbf{B}_1 \hookrightarrow \mathbf{D}$ and a $g_2 : \mathbf{B}_2 \to \mathbf{D}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.



Theorem (Ch. Pech, M. Pech; 2016)

(1) $\mathcal{U}$ is countable & homomorphism-homogeneous $\implies \mathrm{Age}(\mathcal{U})$ has the HAP.

(2) If $\mathscr{C}$ is a class of finite σ-structures, closed under isomorphisms and substructures, with countably many isomorphism types, having the JEP and HAP then there exists a countable homomorphism-homogeneous $\mathcal{U}$ with age $\mathscr{C}$.

We are onto something!

Let $\mathscr{C}$ be a class of finite σ-structures and $n > 0$.

$\mathscr{C}$ has the **nPAP** if for all $i \in \{1, \ldots, n\}$, $\mathbf{A}_i, \mathbf{B}_i, \mathbf{C} \in \mathscr{C}$, $f_1 \colon \mathbf{A} \to \mathbf{C}$ an n-ary polymorphism and $f_2 \colon \mathbf{A} \hookrightarrow \mathbf{B}$, where $\mathbf{A} \leqslant \prod_{i=1}^{n} \mathbf{A}_i$ and $\mathbf{B} \leqslant \prod_{i=1}^{n} \mathbf{B}_i$, there exists a $\mathbf{D} \in \mathscr{C}$, $g_1 \colon \mathbf{C} \hookrightarrow \mathbf{D}$ and an n-ary polymorphism $g_2 \colon \mathbf{B} \to \mathbf{D}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

$$\mathscr{C} \text{ has the } \mathbf{PAP} \iff \mathscr{C} \text{ has the } n\text{PAP}, \forall n > 0.$$

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{g_2} & \mathbf{D} \\ {\scriptstyle f_2}\uparrow & & \uparrow{\scriptstyle g_1} \\ \mathbf{A} & \xrightarrow{f_1} & \mathbf{C} \end{array}$$

$$\mathrm{Age}(\mathcal{U}) \text{ has the PAP} \iff \mathrm{Age}(\mathcal{U}^n) \text{ has the HAP, for all } n > 0.$$

Fraïssé at last!

Theorem

(1) $\mathcal{U}$ is countable & polymorphism-homogeneous $\implies$ $\mathrm{Age}(\mathcal{U})$ has the PAP.

(2) If $\mathscr{C}$ is a class of finite σ-structures, closed under isomorphisms and substructures, with countably many isomorphism types, having the JEP and PAP then there exists a countable, polymorphism-homogeneous $\mathcal{U}$ with age $\mathscr{C}$.

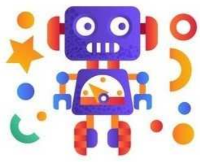
Sketch of the proof - (2):

- $\mathscr{C}$ has PAP $\implies$ $\mathscr{C}$ has the HAP;
- $\exists$ a countable homomorphism-homogenous $\mathcal{U}$, such that $\mathrm{Age}(\mathcal{U}) = \mathscr{C}$.
- Thus, $\mathrm{Age}(\mathcal{U})$ has the PAP.
- Overall, $\mathcal{U}$ is polymorphism-homogenous, too!

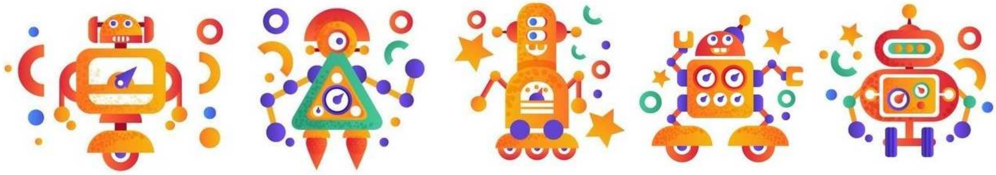
□

Bring in the robots!

Employment procedure



Robot 0 – makes sure that $\mathrm{Age}(\mathcal{U}) = \mathscr{C}$



$\dots$

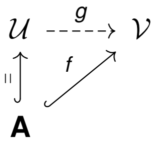
$\forall n > 0$: **Robot n** – secures *n*-polymorphism-homogeneity

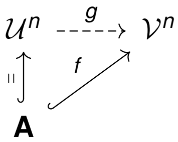
How similar do they get?

$\mathcal{U},\mathcal{V}$ – σ-structures, with $\mathrm{Age}(\mathcal{U}) = \mathrm{Age}(\mathcal{V})$

$\mathcal{U}$ and $\mathcal{V}$ are:

- **H-equivalent** if every $f\colon \mathbf{A} \hookrightarrow \mathcal{V}$ from a finite $\mathbf{A} \leqslant \mathcal{U}$ into $\mathcal{V}$ can be extended to $g\colon \mathcal{U} \to \mathcal{V}$, and vice versa.





- **P-equivalent** if $\forall n > 0$, every $f\colon \mathbf{A} \hookrightarrow \mathcal{V}^n$ from a finite $\mathbf{A} \leqslant \mathcal{U}^n$ into $\mathcal{V}^n$ can be extended to $g\colon \mathcal{U}^n \to \mathcal{V}^n$, and vice versa.

$\mathcal{U}$ and $\mathcal{V}$ are *P*-equivalent $\iff$ $\mathcal{U}^n$ and $\mathcal{V}^n$ are *H*-equivalent, $\forall n > 0$

Can we tell them apart?

$\mathcal{U}, \mathcal{V}$ – countable σ-structures

Proposition (Ch. Pech, M. Pech; 2016)

(1) $\mathcal{U}, \mathcal{V}$ are *H*-equivalent $\implies$ $\mathcal{U}$ is homomorphism-homogeneous iff $\mathcal{V}$ is.

(2) $\mathcal{U}, \mathcal{V}$ are homomorphism-homogeneous & $\mathrm{Age}(\mathcal{U}) = \mathrm{Age}(\mathcal{V})$ $\implies$ $\mathcal{U}, \mathcal{V}$ are *H*-equivalent.

Proposition

(1) $\mathcal{U}, \mathcal{V}$ are *P*-equivalent $\implies$ $\mathcal{U}$ is polymorphism-homogeneous iff $\mathcal{V}$ is.

(2) $\mathcal{U}, \mathcal{V}$ are polymorphism-homogeneous & $\mathrm{Age}(\mathcal{U}) = \mathrm{Age}(\mathcal{V})$ $\implies$ $\mathcal{U}, \mathcal{V}$ are *P*-equivalent.

Thank you for your attention!