Congruence lattices of connected monounary algebras

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Introduction

- the system of all congruences of an algebra (A, F) forms a lattice, denoted $\mathrm{Con}(A, F)$
- the system of all congruence lattices of all algebras with the base set A forms a lattice $\mathcal{E}_A$
- we deal with meet-irreducibility in $\mathcal{E}_A$ for a given finite set A
- all meet-irreducible elements of $\mathcal{E}_A$ are congruence lattices of monounary algebras

Introduction

- **open problem:** characterization of all $\wedge$-irreducible elements in $\mathcal{E}_A$
- some types of meet-irreducible congruence lattices were described by Jakubíková-Studenovská, Pöschel and Radeleczki (2018); and by Janičková and Jakubíková-Studenovská (2018)
- today, we present **necessary and sufficient condition** under which $\mathrm{Con}(A, f)$ is meet-irreducible in the case when (A, f) is **connected algebra** (i.e., it contains only one component).

Preliminary

(A, f) - monounary algebra, A is finite set

- operation $f \in A^A$ is called **trivial**, if it is identity of constant
- an element $x \in A$ is called **cyclic** if it belongs to cycle, otherwise it is called **noncyclic**
- let $t_f(a) := \min\{n \in \mathbb{N}_0 : f^n(a)$ is cyclic $\}$
- for $k \in \mathbb{N}$ we denote $C_k := \{x \in A : t_f(x) = k\}$
- for $x, y \in A$ let $\theta_f(x, y)$ be the smallest congruence on (A, f) such that $(x, y) \in \theta_f(x, y)$

Preliminary

Definition

A monounary algebra (A, f) is called *connected*, if it contains a single cycle.

Definition

A monounary algebra (A, f) is called a *permutation-algebra* if the operation f is a permutation on A.

Definition

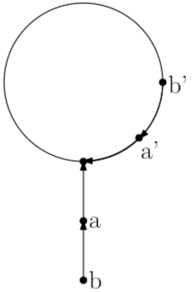
A monounary algebra (A, f) is called a *permutation-algebra with short tails* if $f(x)$ is cyclic for every $x \in A$

Preliminary

Definition

For every element $x \in A$ such that $t_f(x) = t$, there is a single cyclic element $y \in A$ such that $f^t(x) = f^t(y)$. We will call this element a *colleague* of x and we denote it x'.

Example:



Preliminary

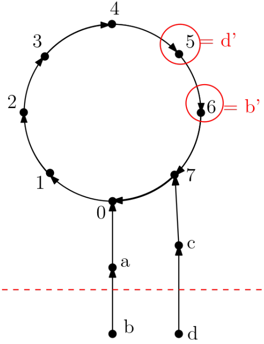
If (A, f) is either a permutation-algebra with short tails or its cycle contains at most 2 elements, then the necessary and sufficient conditions under which $\mathrm{Con}(A, f)$ is meet-irreducible were already proved.

It remains to study monounary algebras (A, f) such that (A, f) contains at least 3 cyclic elements, or it contains an element x such that $f(x)$ is noncyclic.

Definition

Let (A, f) be a connected algebra with the cycle C such that $|C| = n, n \geq 3$. Next, let $C = \{0, 1, \ldots, n-1\}$ where $f(0) = 1, f(1) = 2, \ldots, f(n-1) = 0$. We say that a cyclic element c is *covered* if there exists a noncyclic $x \in A \setminus C_1$ such that $c = x'$.

Example:

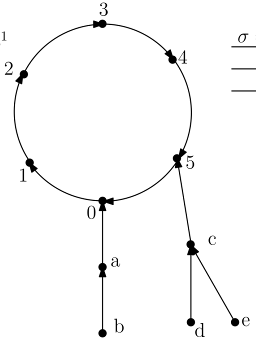


Preliminary

Definition

Further, we denote the canonical decomposition of the number of elements of C as $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$; the numbers $p_l^{\alpha_l}$ for $l \in \{1, \ldots, k\}$ are said to be elementary divisors of n. The cycle C is called *covered* if each equivalence class modulo σ, where σ is an elementary divisor of n, contains at least one covered $c \in C$.

$n = 2^1 \cdot 3^1$



$\sigma = 3$	0, 3	3 is covered by d (and by e)
	1, 4	4 is covered by b
	2, 5	nothing is covered

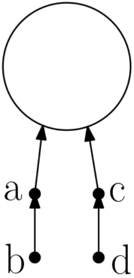
$\Rightarrow$ cyclus is not covered

Necessary and sufficient conditions

Notation

Consider the following two conditions:
$(*1)$ *the set* C *is covered,*
$(*2)$ *there exist distinct noncyclic elements* $a, b, c, d \in A$ *such that* $f(a), f(c) \in C$ *and* $f(b) = a, f(d) = c$.

Condition ($*$2):



Necessary condition

First, we prove that if either of the conditions $(*1), (*2)$ is not satisfied, then $\mathrm{Con}(A, f)$ is $\wedge$-reducible.

Necessary condition

Assume that $(*1)$ is not satisfied and that (A, f) is not a permutation algebra. This yields that there are $l \in \{1, \ldots, k\}$ and $i \in C$ such that $c \equiv i \pmod{p_l^{\alpha_l}}$ fails to hold for any covered element $c \in C$. Without loss of generality, let $i = 0$.

Now we set $p = p_l$, $\alpha = \alpha_l$, $\beta = \frac{n}{p^\alpha}$, $j = p^{\alpha-1} \cdot \beta$.

We define the following operations on A:

$$g_3(x) := x' + p^\alpha,$$
$$g_4(x) := x' + \beta$$

for each $x \in A$, and

$$g(x) := \begin{cases} x' + j + 1 & \text{if } x \in C \cup C_1, p^\alpha \text{ divides } x', \\ f(x) & \text{otherwise.} \end{cases}$$

We proved that $\mathrm{Con}(A, f) = \mathrm{Con}(A, g_3) \cap \mathrm{Con}(A, g_4) \cap \mathrm{Con}(A, g)$.

Necessary condition

Proposition

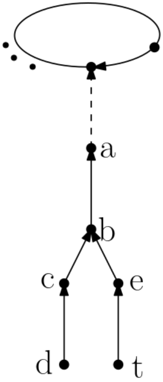
If (A, f) *does not satisfy the condition* $(*1)$*, then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-reducible.*

Necessary condition

Now assume that the condition $(*2)$ is not valid and that C_2 is nonempty.

Let k be the least positive integer such that there exist distinct noncyclic elements $a, b, c, d, e, t \in A$ such that $b \in C_k$, $f(b) = a$, $f(c) = f(e) = b$, $f(d) = c$, $f(t) = e$. If such k does not exist, then put $k = 1$ and $a = f(b)$ for $c \in C_2, b = f(c)$.

Example:



Necessary condition

First we will investigate the case $k > 1$.
We define the operations g_5 and g_6 on A by putting

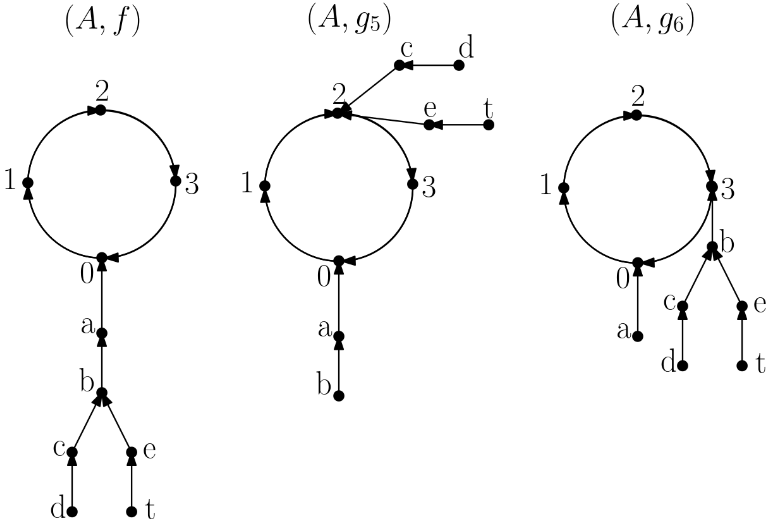
$$g_5(x) := \begin{cases} b' & \text{if } f(x) = b, \\ f(x) & \text{otherwise,} \end{cases}$$

$$g_6(x) := \begin{cases} a' & \text{if } f(x) = a, \\ f(x) & \text{otherwise.} \end{cases}$$

Analogously like before, $\mathrm{Con}(A, f) = \mathrm{Con}(A, g_5) \cap \mathrm{Con}(A, g_6)$, hence $\mathrm{Con}(A, f)$ is $\wedge$-reducible.

Necessary condition

Example:



Necessary condition

Now we deal with the case $k = 1$.
Let g_7 and g_8 be the following operations on A:

$$g_7(x) := \begin{cases} b' & \text{if } f(x) = b, \\ f(x) & \text{otherwise,} \end{cases}$$

$$g_8(x) := \begin{cases} b & \text{if } f(x) \in C, \\ b' & \text{otherwise.} \end{cases}$$

Analogously like before, $\mathrm{Con}(A, f) = \mathrm{Con}(A, g_7) \cap \mathrm{Con}(A, g_8)$, hence $\mathrm{Con}(A, f)$ is $\wedge$-reducible.

Necessary condition

Proposition

If (A, f) *does not satisfy the condition* $(*2)$*, then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-reducible.*

Sufficient condition

Next, we suppose that the condition $(*1)$ and the condition $(*2)$ are satisfied. Our aim is to prove that in this case, $\mathrm{Con}(A, f)$ is $\wedge$-irreducible.

Sufficient condition

- By way of contradiction, suppose that $\mathrm{Con}(A, f)$ is $\wedge$-reducible. Then there exists a set G of nontrivial operations on A such that $|G| \geq 2$ and

$$\mathrm{Con}(A, f) = \bigcap_{g \in G} \mathrm{Con}(A, g),\ \mathrm{Con}(A, f) \neq \mathrm{Con}(A, g) \text{ for } g \in G.$$

- This implies that for every $x, y \in A$: $(g(x), g(y)) \in \theta_f(x, y)$.

Sufficient condition

- According to condition $(*2)$, there exist distinct noncyclic elements $a, b, c, d \in A$ such that $f(a), f(c) \in C$ and $f(b) = a$, $f(d) = c$.
- This and $\forall x, y \in A : (g(x), g(y)) \in \theta_f(x, y)$ yield some conditions for $g(a), g(b), g(c), g(d)$ and $g(a'), g(b'), g(c'), g(d')$.
- We proved that $g(x) = f(x)$ for each $x \in A$, which is a contradiction with $\mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g)$.

Sufficient condition

Proposition

If the conditions $(*1)$ *and* $(*2)$ *are valid, then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-irreducible.*

Main result

Theorem

Let (A, f) be a connected monounary algebra with at least 3 cyclic elements. Then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible if and only if the conditions $(*1)$ and $(*2)$ are satisfied.

References

- Jakubíková-Studenovská, D., Janičková, L.: **Meet-irreducible congruence lattices.** Algebra Universalis. 79(4), (2018).
- Jakubíková-Studenovská, D., Pöschel, R., Radeleczki, S.: **The lattice of congruence lattices of algebra on a finite set.** Algebra Universalis. 79(2), (2018).
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Thank you for your attention.