TECHNISCHE UNIVERSITÄT WIEN

All centralising monoids given by conservative majority operations on $\{0, 1, 2, 3\}$

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A Galois correspondence

$\mathcal{O}_A^{(n)} = A^{A^n} = \{ f \mid f\colon A^n \longrightarrow A\}$ $\qquad \mathcal{O}_A = \bigcup_{n\in\mathbb{N}_+} A^{A^n}$

Commutation of $f \in \mathcal{O}_A^{(m)}$ with $g \in \mathcal{O}_A^{(n)}$

$$g \perp f \;:\Longleftrightarrow\; g \in \operatorname{Hom}(\langle A; f\rangle^n; \langle A; f\rangle)$$

Centraliser of $F \subseteq \mathcal{O}_A$

$$\begin{aligned} F^* &= \{ g \in \mathcal{O}_A \mid \forall f \in F\colon g \perp f\} \\ &= \bigcup_{n\in\mathbb{N}_+} \operatorname{Hom}(\langle A; F\rangle^n; \langle A; F\rangle) \end{aligned}$$

(that's a clone)

$$F^{**} = (F^*)^* \supseteq \langle F\rangle_{\mathcal{O}_A} \supseteq F$$

(bicentraliser)

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Centraliser of $F \subseteq \mathcal{O}_A$

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$$= \mathrm{Pol}_A F^\bullet$$

(that's a clone)

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(bicentraliser)

Unary commuting operations

$F \subseteq \mathcal{O}_A$

$F^{*(1)} = \mathrm{Hom}(\langle A; F\rangle; \langle A; F\rangle) = \mathrm{End}(\langle A; F\rangle)$.

unary part of a centraliser

= endomorphism monoid of an algebra

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unary part of a centraliser

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$s \in \mathcal{O}_A^{(1)}$

$$s \in F^{*(1)} \iff \forall f \in F\ \underbrace{\forall \mathbf{x} \in A^{\operatorname{ar} f}\colon s(f(\mathbf{x})) = f(s \circ \mathbf{x})}_{s \perp f}$$

Centralising monoids + observations

For $M \subseteq \mathcal{O}_A^{(1)}$ TFAE

1. M is a centralising monoid on A

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Note: $M \subseteq F^*$

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Maximal centralising monoids

Maximal centralising monoid M... $|A| < \aleph_0$

...coatom in the lattice of centralising monoids on A

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Minimal functions

= a minimum arity generator f of a minimal clone;
by Rosenberg's Theorem, f is

1. special unary function $f \in \mathcal{O}_A^{(1)}$
2. $f \in \mathcal{O}_A^{(2)}$, idempotent: $f(x, x) \approx x$
3. $f \in \mathcal{O}_A^{(3)}$ minority $f(x, y, z) \approx x \oplus y \oplus z$
4. $f \in \mathcal{O}_A^{(3)}$ majority $f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \approx x$
5. $f \in \mathcal{O}_A^{(k)}$ proper semiprojection, $3 \leq k \leq |A|$

(Maximal) centralising monoids for $|A| \geq 3$

All centralising monoids for $|A| = 3$

- ISMVL 2011 Machida, Rosenberg
- ISMVL 2015 Goldstern, Machida, Rosenberg
- 192 monoids identified, 10 maximal ones
  for maximals: only unary and majority witnesses needed

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Next step: maximal ones for $|A| = 4$

. . . **much** harder (combinatorial explosion)

- witnesses $\leftrightarrow$ types of minimal functions, one at a time
- MB, 2021$^{(?)}$: all centralising monoids with majority witn.
  1 715 monoids, 147 maximal ones (among the 1 715)

In this talk. . .

Again majority operations on $\{0, 1, 2, 3\}$ as witnesses

. . . all centralising monoids with conservative majority witn.

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$f \in \mathcal{O}_A$ conservative $\iff$ $\forall B \subseteq A\colon \quad B \leq \langle A; f \rangle$

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$f \in \mathcal{O}_A$ conservative $\iff$ $\forall B \subseteq A\colon \quad B \leq \langle A; f\rangle$

Why conservative?

- maximal centralising monoids $\leftrightarrow$ minimal functions
- minimal clones gen. by majority operations:
  described for $|A| \leq 4$ Csákány 1983, Waldhauser 2000
- minimal clones gen. by conservative majority operations:
  described for $|A| < \aleph_0$! Csákány 1986

  (all subsets are subuniverses, restriction is a clone hom.)

Results for $A = \{0, 1, 2, 3\}$

Centralising monoids M with conservative majority witnesses

- $\mathcal{O}_A^{(1)}$ + 720 proper centralising monoids (cons. maj. witn.)
  42 % of the 1 715
- all 720: $M = \{f\}^{*(1)}$ (∃ single cons. maj. witness)
  242 conjugacy classes of witnesses
- more efficiently: $\exists G$ of 226 cons. maj. operations:
  $$\forall \text{monoid } M\, \exists F \subseteq G: \quad M = F^{*(1)}$$
  (all witnesses drawn from a set G of 226 cons. maj. ops., with 52 conjugacy types)

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Maximal centralising monoids with cons. maj. witn.

- 107, all among the 147 maximal monoids with maj. witn.,
- 12 maximal monoids up to conjugacy,
- 10–12 up to isomorphism

Using the Galois connection (aka formal context)

all unary ops. $\mathcal{O}_4^{(1)}$

$4^4 = 256$

	×		×	×	×	×	⋯		×	×
		×	×	×	×	×	⋯			
	×	×	×	×	×	×	⋯			×
					⋮					
all cons. maj.			×	×	×	×			×	
	×		×	×				×		
	×	×		×					×	
			×		×	×		×		×

Using the Galois connection (aka formal context)

$\forall\, M$

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$4^4 = 256$

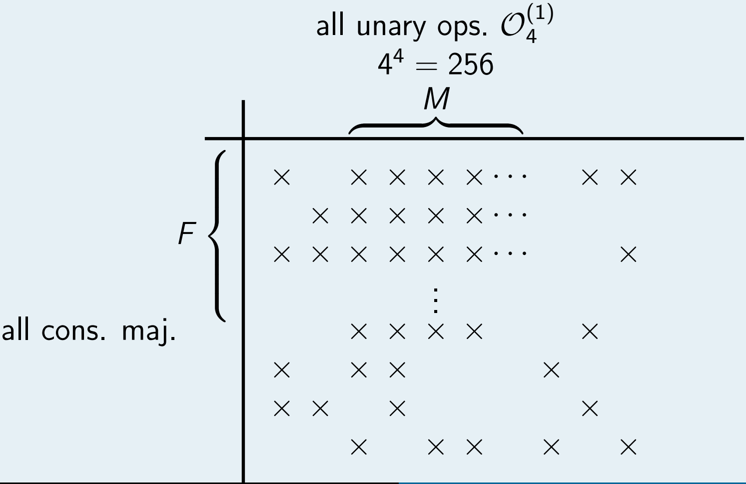
M (brace over the middle columns)

	×		×	×	×	×	⋯		×	×
		×	×	×	×	×	⋯			
	×	×	×	×	×	×	⋯			×
					⋮					
all cons. maj.			×	×	×	×			×	
	×		×	×				×		
	×	×		×					×	
			×		×	×		×		×

How to get there

Using the Galois connection (aka formal context)

$\forall M \, \exists F \subseteq \mathrm{Maj}_A \cap \mathrm{Pol}_A \mathfrak{P}(A)$:



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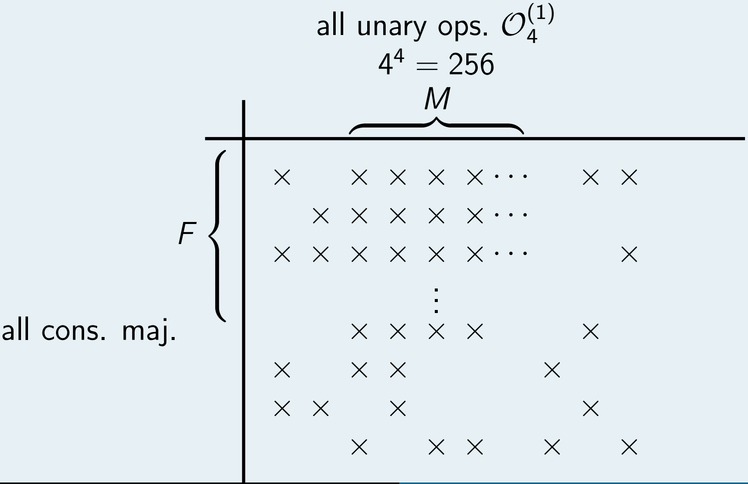

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			M							
F	$\times$		$\times$	$\times$	$\times$	$\times$	$\cdots$		$\times$	$\times$
F		$\times$	$\times$	$\times$	$\times$	$\times$	$\cdots$			
F	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\cdots$			$\times$
F					$\vdots$					
all cons. maj.			$\times$	$\times$	$\times$	$\times$			$\times$	
	$\times$		$\times$	$\times$				$\times$		
	$\times$	$\times$		$\times$					$\times$	
			$\times$		$\times$	$\times$		$\times$		$\times$

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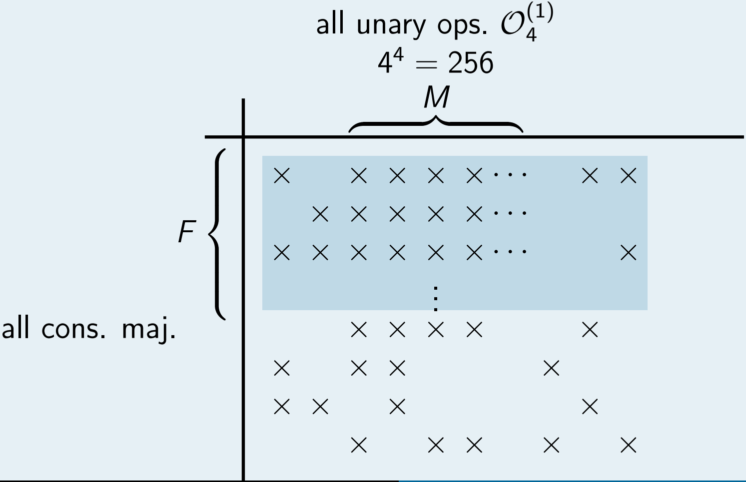

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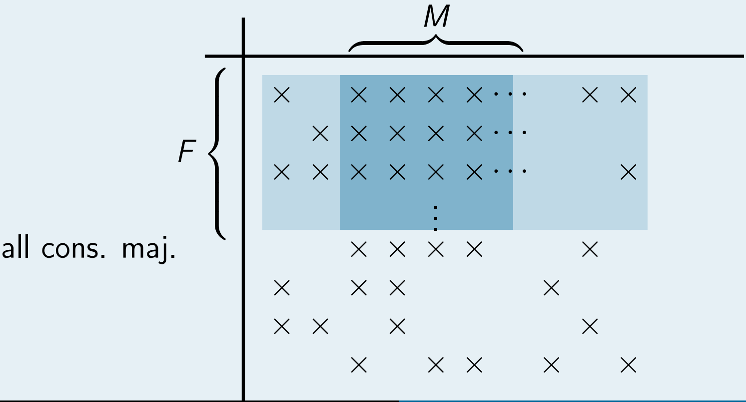

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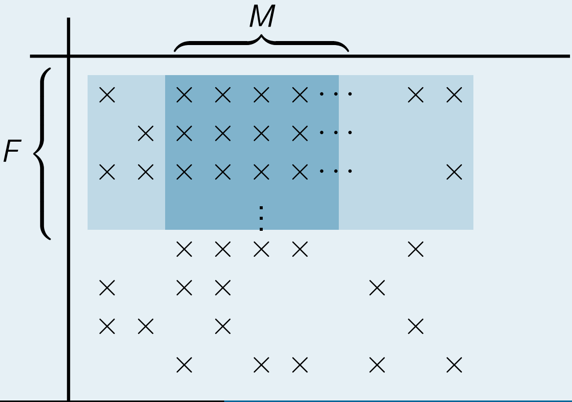
Just compute arbitrary intersections!

all unary ops. $\mathcal{O}_4^{(1)}$

$4^4 = 256$

all cons. maj.



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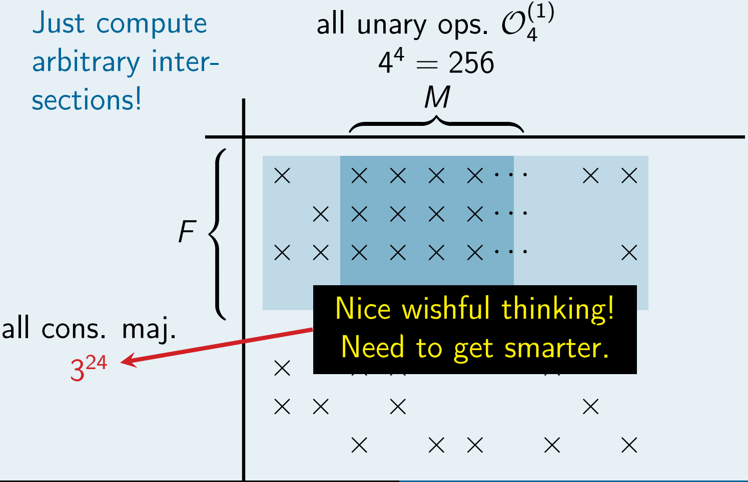


How to get there

Using the Galois connection (aka formal context)

all unary ops. $\mathcal{O}_4^{(1)}$

$4^4 = 256$

trivial

s

	×		×	×	×	×	⋯		×	×	×	×
		×	×	×	×	×	⋯				×	×
	×	×	×	×	×	×	⋯			×	×	×
					⋮						×	×
all cons. maj.			×	×	×	×			×		×	×
3^{24}	×		×	×				×			×	×
	×	×		×					×		×	×
			×		×	×		×		×	×	×

Using the Galois connection (aka formal context)

all unary ops. $\mathcal{O}_4^{(1)}$

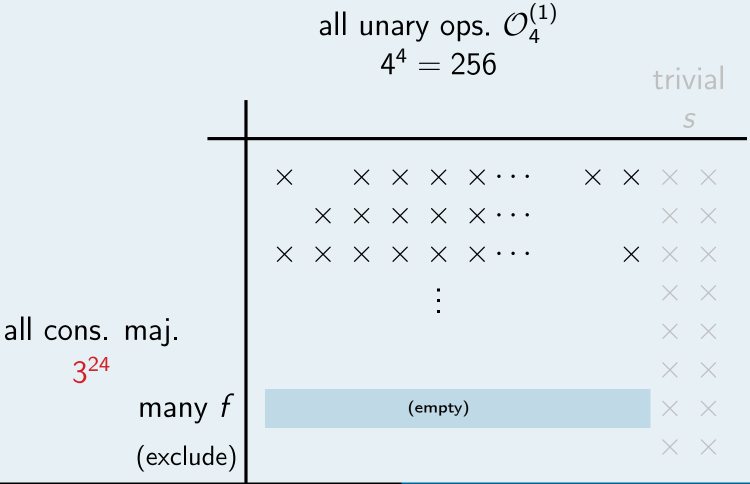
$4^4 = 256$

all cons. maj.

3^{24}

										trivial s	
×		×	×	×	×	···		×	×	×	×
	×	×	×	×	×	···				×	×
×	×	×	×	×	×	···			×	×	×
				⋮						×	×
		×	×	×	×			×		×	×
×		×	×				×			×	×
×	×		×					×		×	×
		×		×	×		×		×	×	×

Using the Galois connection (aka formal context)



How to get there

Using the Galois connection (aka formal context)

$\forall s$ nontrivial

all unary ops. $\mathcal{O}_4^{(1)}$

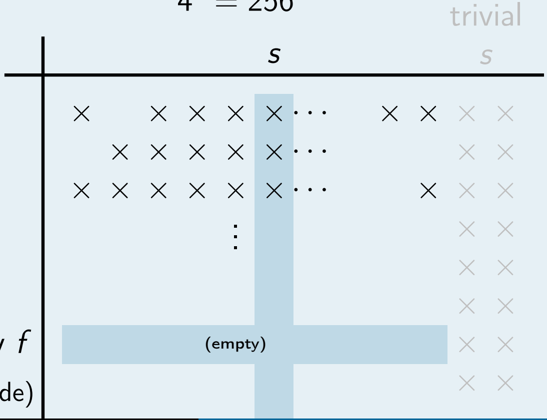
$4^4 = 256$

all cons. maj.

3^{24}

many f

(exclude)



Using the Galois connection (aka formal context)

$\forall s$ nontrivial $\forall$ cons. $f \in \{s\}^* \cap \mathrm{Maj}_A$:

all unary ops. $\mathcal{O}_4^{(1)}$

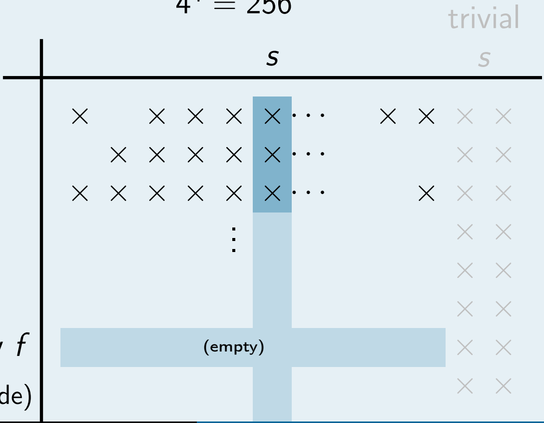
$4^4 = 256$

all cons. maj.

3^{24}

many f

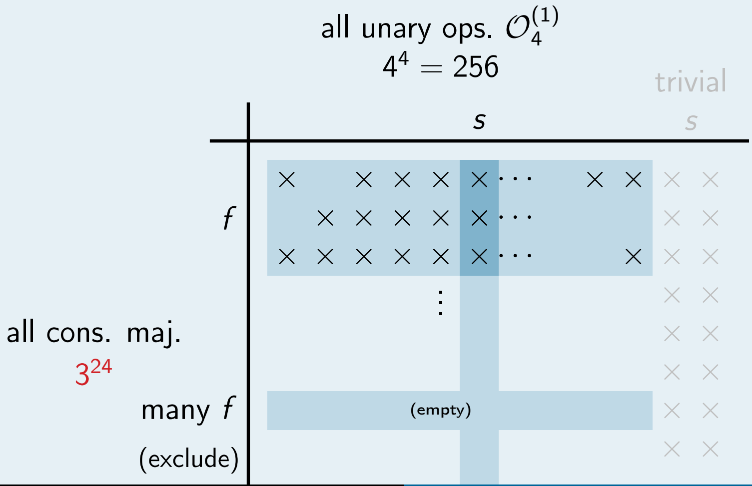
(exclude)



How to get there

Using the Galois connection (aka formal context)

$\forall s$ nontrivial $\forall$ cons. $f \in \{s\}^* \cap \mathrm{Maj}_A$: store $\{f\}^{*(1)}$



How to get there

Using the Galois connection (aka formal context)

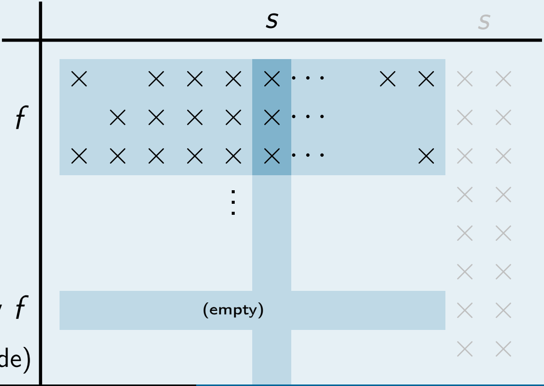
$\forall s$ nontrivial $\forall$ cons. $f \in \{s\}^* \cap \mathrm{Maj}_A$: store $\{f\}^{*(1)}$

Now compute arbitrary intersections!

all unary ops. $\mathcal{O}_4^{(1)}$

$4^4 = 256$

trivial



all cons. maj.

3^{24}

many f

(exclude)

The end. . .

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. . . of the COVID pandemic hopefully comes soon.

The end...

...of the COVID pandemic hopefully comes soon.

This is just the end of my talk.

- Any remarks / questions are welcome.
- Thank you for your attention.