Oligomorphic Clones (Part 1 of 2)

Manuel Bodirsky

Institut für Algebra, TU Dresden

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Outline

General goal:

results for algebras on finite domains $\longrightarrow$ algebras with infinite domains

Tutorial part 1:

- Goal fails badly in general
- Fruitful assumption: oligomorphicity
- Examples
- Fundamental results

Tutorial part 2:

- More advanced results (Barto, Mottet, Opršal, Pinsker, . . .)
- Applications
    - Constraint satisfaction
    - Finite model theory, database theory
    - Theory of relation algebras (e.g., work of Andréka, Hirsch, Hodkinson, Maddux, . . .), network satisfaction

Pol-Inv

$\underline{A} = (A; f_1, f_2, \ldots)$: algebra.
$\mathcal{C} = \mathrm{Clo}(\underline{A})$: clone of term operations of A.

Definition. $f\colon A^k \to A$ preserves $R \subseteq A^m$ if for all $a_1, \ldots, a_k \in R$

$$\underbrace{f(a_1, \ldots, a_k)}_{\text{computed componentwise}} \in R$$

$\mathcal{C}$: clone on A.

$\mathrm{Inv}(\mathcal{C})$: set of all relations preserved by every $f \in \mathcal{C}$.

$\mathcal{R}$: set of relations on A.

$\mathrm{Pol}(\mathcal{R})$: set of all operations preserving every $R \in \mathcal{R}$.

Fact. If A is finite, then

$$\mathrm{Pol}(\mathrm{Inv}(\mathcal{C})) = \mathcal{C}.$$

Infinite Example

$s_n \colon \mathbb{Z} \to \mathbb{Z}$ given by $x \mapsto x + n$
$t \colon \mathbb{Z} \to \mathbb{Z}$: the transposition $(0, 1)$.

$$\underline{A} := (\mathbb{Z}; s_1, s_{-1}, t)$$

$\text{Clo}(\underline{A})$. Countable.
$\text{Pol}(\text{Inv}(\text{Clo}(\underline{A})))$: contains all injections $\mathbb{Z} \to \mathbb{Z}$. Uncountable.

A^{A^k} equipped with product topology (for A discrete).
for each $a_0, a_1, \dots, a_k \in A^m$ basic open set:

$$B_{a_0, a_1, \dots, a_k} := \{ f \colon A^k \to A \mid f(a_1, \dots, a_k) = a_0 \}$$

$\bigcup_{k \geq 1} A^k \to A$: equipped with sum-space topology

Fact.

$$\text{Pol}(\text{Inv}(\mathcal{C})) = \overline{\mathcal{C}}$$

Inv-Pol

$\mathfrak{A} = (A; R_1, R_2, \ldots)$: relational structure.

$\text{Pol}(\mathfrak{A}) := \text{Pol}(\{R_1, R_2, \ldots\})$

$\text{Inv}(\text{Pol}(\mathfrak{A}))$

Theorem (Geiger'68, Bodnarčuk, Kalužnin, Kotov, Romov'69):
If A is finite, then

$$\text{Inv}(\text{Pol}(\mathfrak{A})) = \langle \mathfrak{A} \rangle_{pp} := \{R \mid R \text{ has primitive positive (pp) definition in } \mathfrak{A}\}.$$

Primitive positive formulas are of the form:

$$\exists x_1, \ldots, x_n \left(\underbrace{\psi_1 \wedge \cdots \wedge \psi_m}_{\psi_1, \ldots, \psi_m \text{ atomic formulas}} \right)$$

Application: Complexity of Constraint Satisfaction.

Inv-Pol on Infinite Domains

$$\mathfrak{A} = \big(\mathbb{Z}; \{0\}, \{(x,y) \mid y = x+1\}, \{(a,b,c,d) \mid a = b \Rightarrow c = d\}\big)$$

$\text{Pol}(\mathfrak{A}) = \{\text{all projections}\}$

$$\text{Inv}(\text{Pol}(\mathfrak{A})) = \{\text{all relations on } A\}$$

General description of $\text{Inv}(\text{Pol}(\mathfrak{A}))$: Szabó, Geiger, Pöschel, using infinitary relations.

Stronger fact. If $\mathfrak{A}$ is countable, then $\text{Inv}(\text{Pol}(\mathfrak{A}))$ equals smallest set of (finitary!) relations containing the relations of $\mathfrak{A}$ that is

- closed under pp definitions,
- infinite intersections, and
- direct unions.

Observation: If for each arity only finitely many pp definable relations:

$$\text{Inv}(\text{Pol}(\mathfrak{A})) = \langle \mathfrak{A} \rangle_{\text{pp}}.$$

Oligomorphic Clones

$\mathcal{C}^{(1)}$: the unary operations in $\mathcal{C}$.
$\mathcal{G} := \{u \in \mathcal{C}^{(1)} \mid \exists v \in \mathcal{C}^{(1)}$ s.t. $u \circ v = v \circ u = \text{id}\}$: a permutation group.
Note: If $\mathcal{C} = \text{Pol}(\mathfrak{A})$ then $\mathcal{G} = \text{Aut}(\mathfrak{A})$.

Definition (Oligomorphicity)

Let $\mathcal{G}$ be a permutation group on a countably infinite set A.
$\mathcal{G}$ is oligomorphic if for every $m \in \mathbb{N}$ the action $\mathcal{G} \curvearrowright A^m$ has finitely many orbits, i.e., finitely many sets of the form

$$\{u(a) \mid u \in \mathcal{G}\} \qquad \text{for } a \in A^m.$$

Clear: $\text{Aut}(\mathfrak{A})$ oligomorphic $\Rightarrow$ finitely many pp definable relations in $\mathfrak{A}$.

$$\text{Aut}(\mathfrak{A}) \text{ oligomorphic } \Leftrightarrow (\mathfrak{A})_{\text{fo}} \text{ has finitely many relations for every fixed arity}$$
$$\Leftrightarrow \text{Aut}(\mathfrak{A}) \curvearrowright \binom{A}{m} \text{ has finitely many orbits for every } m \in \mathbb{N}$$

Examples

A clone $\mathscr{C}$ is oligomorphic if it contains an oligomorphic permutation group.
An algebra $\underline{A}$ is oligomorphic if $\text{Clo}(\underline{A})$ is oligomorphic.

- $\mathcal{G} = \text{Sym}(\mathbb{N})$.
  $\mathcal{C} = \text{Pol}(\mathbb{N}; \neq, \{(a,b,c,d) \mid a = b \Rightarrow c = d\})$.
  Uncountably many clones $\mathscr{C}$ with $\mathcal{G} \subseteq \mathscr{C}^{(1)}$ (B., Chen, Pinsker'10).
- $\mathcal{G} = \text{Aut}(\mathbb{Q}; <)$.
  $\text{Aut}(\mathbb{Q}; <) \curvearrowright \binom{\mathbb{Q}}{m}$ has one orbit, for every m.
  $\mathcal{C} = \text{Pol}(\mathbb{Q}; \{(u,v,w) \mid u > v \vee u > w\})$.
- Non-example: $\text{Pol}(\mathbb{Z}; <)$.
  $\text{Aut}(\mathbb{Z}; <) \curvearrowright \binom{\mathbb{Z}}{1}$: 1 orbit.
  $\text{Aut}(\mathbb{Z}; <) \curvearrowright \binom{\mathbb{Z}}{2}$: infinitely many orbits!

Countable Categoricity

Theorem (Engeler, Ryll-Nardzewski, Svenonius).

Let $\mathfrak{A}$ be a countable structure. Then $\text{Pol}(\mathfrak{A})$ is oligomorphic if and only if $\mathfrak{A}$ is ω-categorical, i.e., all countable models of the first-order theory of $\mathfrak{A}$ are isomorphic.

Examples.

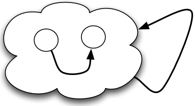
- $(\mathbb{Q}; <)$
  (Cantor: all countable dense unbounded linear orders are isomorphic)
- the countable atomless Boolean algebra,
- the Rado graph := the (up to isomorphism unique) countable model of the almost-sure theory of $G_{n,1/2}$

Proposition (B.+Nešetřil'03). If $\mathfrak{A}$ is a countable ω-categorical structure, then

$$\text{Inv}(\text{Pol}(\mathfrak{A})) = \langle \mathfrak{A} \rangle_{\text{pp}}.$$

Homogeneity

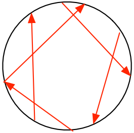
A structure $\mathfrak{A}$ is homogeneous iff every isomorphism between finitely generated substructures of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$.



Observation. Every homogeneous structure with a finite relational signature has an oligomorphic polymorphism clone.

'finitely homogeneous structures'

Examples: $(\mathbb{N}; \neq)$, $(\mathbb{Q}; <)$, $S(2)$, the 'dense local order'.



Homogeneous structures as Fraïssé-limits:
if $\mathcal{C}$ is a class of finite structures such that

- $\mathcal{C}$ is closed under isomorphism and substructures
- $\mathcal{C}$ has the amalgamation property

then there exists an (up to isomorphism unique) homogeneous structure $\mathfrak{L}$ such that $\mathcal{C} = \text{Age}(\mathfrak{L}) := \{\mathfrak{A} \text{ finite} \mid \mathfrak{A} \hookrightarrow \mathfrak{L}\}$.

Congruences

Definition. A congruence of an algebra $\underline{A} = (A; f_1, f_2, \dots)$ is an equivalence relation in $\text{Inv}(\{f_1, f_2, \dots\})$.

Observations: If $\underline{A}$ is oligomorphic, then

- $\underline{A}/{\sim}$ is oligomorphic.
- $\underline{A}$ has finitely many congruences.
- $\underline{A}$ has a unique coarsest congruence with finite classes.
- $\underline{A}$ has a unique finest congruence with finitely many classes.
- May also have congruences with infinitely many infinite classes.

Birkhoff's Theorem

Let $\underline{A}$ be an oligomorphic algebra.

Observations.

- the variety $\mathrm{HSP}(\underline{A})$ contains algebras that are not oligomorphic.
- all algebras in the pseudo-variety $\mathrm{HSP}^{\mathrm{fin}}(\underline{A})$ are oligomorphic.

Theorem (B.+Pinsker'15). Let $\underline{A}, \underline{B}$ be oligomorphic algebras with the same signature. Then the following are equivalent:

- $\underline{B} \in \mathrm{HSP}^{\mathrm{fin}}(\underline{A})$.
- the natural homomorphism $\mathrm{Clo}(\underline{A}) \to \mathrm{Clo}(\underline{B})$ exists and is continuous.

Corollaries:

- There is an isomorphism $\mathrm{Pol}(\mathfrak{A}) \to \mathrm{Pol}(\mathfrak{B})$ which is a homeomorphism if and only if $\underline{A}$ and $\underline{B}$ are pp bi-interpretable.
- $\mathrm{Pol}(\mathfrak{A})$ has a continuous homomorphism to $\mathrm{Pol}(K_3)$ if and only if K_3 has a pp interpretation in $\mathfrak{A}$.

Applications for complexity of constraint satisfaction.

Idempotence

An operation f is called idempotent if it satisfies $f(x, \ldots, x) \approx x$.
A clone is called idempotent if all its operations are idempotent.

Observations:

- A clone $\mathcal{C}$ on a set A is idempotent if and only if $\{a\} \in \text{Inv}(\mathcal{C})$ for every $a \in A$.
- Oligomorphic clones on infinite sets are never idempotent.
- Oligomorphic clones may contain interesting idempotent operations:

  e.g.: $\text{Pol}(\mathbb{Q}; <)$ contains $(x, y) \mapsto \min(x, y)$
- $\text{Pol}(\mathbb{N}; \neq)$ contains an operation f that satisfies

  $$f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \approx f(x, x, x)$$

  but no interesting idempotent operation.

Model-Complete Cores

Definition: A structure $\mathfrak{A}$ is called a model-complete core if

$$\overline{\mathrm{Aut}(\mathfrak{A})} = \mathrm{Pol}(\mathfrak{A})^{(1)}.$$

Equivalent: Every first-order formula is equivalent to a pp formula in $\mathfrak{A}$.

Examples: $(\mathbb{N}; \neq)$, $(\mathbb{Q}; <)$, $S(2)$, ...

Non-Examples:

- $\mathfrak{A} := (\mathbb{Q}; \leq)$ has constant polymorphisms,
  but the closure of $\mathrm{Aut}(\mathfrak{A})$ only contains injective functions.
- The Rado graph has non-injective endomorphisms.

Useful Consequence: If $\mathfrak{A}$ is a model-complete core such that for some $g, h\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ there exists $f \in \mathrm{Pol}(\mathfrak{A})$ with

$$f(x_{g(1)}, \ldots, x_{g(n)}) \approx f(x_{h(1)}, \ldots, x_{h(n)})$$

then for every $a \in A^m$, $m \in \mathbb{N}$, the clone $\mathrm{Pol}(\mathfrak{A}, \bar{a})$ also contains such an f.

Homomorphic Equivalence

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are called homomorphically equivalent if there is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and vice versa.

Observation: Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are homomorphically equivalent and $f \in \text{Pol}(\mathfrak{A})$ satisfies

$$f(x_{g(1)}, \dots, x_{g(n)}) \approx f(x_{h(1)}, \dots, x_{h(n)})$$

for some $g, h \colon \{1, \dots, n\} \to \{1, \dots, k\}$, then $\text{Pol}(\mathfrak{B})$ also contains such an f.

Theorem (B.'06): Every countable ω-categorical structure $\mathfrak{A}$ is homomorphically equivalent to a model-complete core structure $\mathfrak{C}$, which is unique up to isomorphism, and ω-categorical.

Example. The Rado graph is homomorphically equivalent to $(\mathbb{N}; \neq)$.
"$(\mathbb{N}; \neq)$ is the model-complete core of the Rado graph"

Siggers Operations

Building on results from Bulatov'05, Hell+Nešetřil'90:

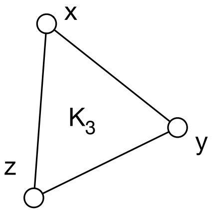
Theorem (Siggers'10).

Let $\mathfrak{A}$ be a finite structure such that $\mathrm{Pol}(\mathfrak{A})$ is idempotent. Then TFAE:

- $\mathrm{Pol}(\mathfrak{A})$ has no clone homomorphism to $\mathrm{Pol}(K_3)$.
- $\mathrm{Pol}(\mathfrak{A})$ contains an operation s such that

$$s(x, y, x, z, y, z) \approx s(z, z, y, y, x, x).$$

All polymorphisms of K_3 are essentially permutations.



Pseudo-Siggers Operations

Theorem (Barto+Pinsker'16).

Let $\mathfrak{A}$ be an ω-categorical model-complete core. Then TFAE:

- for all $n \in \mathbb{N}$, $\bar{a} \in A^n$, there is no continuous clone homomorphism
$$\mathrm{Pol}(\mathfrak{A}, \bar{a}) \to \mathrm{Pol}(K_3).$$
- $\mathrm{Pol}(\mathfrak{A})$ contains operations s, e_1, e_2 such that
$$e_1(s(x,y,x,z,y,z)) \approx e_2(s(z,z,y,y,x,x)).$$

s is called pseudo-Siggers polymorphism of $\mathfrak{A}$.

Exercises

Let $J = (V; E)$ be the line graph of the infinite clique (also called the Johnson graph):

- Vertices: $V := \big\{\{u,v\} \mid u,v \in \mathbb{N}, u \neq v\big\}$
- Edges: $\big\{\{u,v\},\{a,b\}\big\} \in E$ if $|\{u,v,a,b\}| = 3$.

Tasks:

1. Is $\mathrm{Pol}(J)$ oligomorphic?
2. What is the model-complete core of J?
3. Is there a continuous clone homomorphism $\mathrm{Pol}(J) \to \mathrm{Pol}(K_3)$?
4. Is there a clone homomorphism $\mathrm{Pol}(J) \to \mathrm{Pol}(K_3)$?
5. Does J have a Siggers polymorphism?
6. Does J have a pseudo-Siggers polymorphism?

Solutions due: Feb 6, 2020, 18h00.