Dualities for prioritised default bilattices II

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AAA 100
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My coauthors

Slovakia, Sept 2017, where we started our joint study of bilattices

Bilattices: a brief introduction

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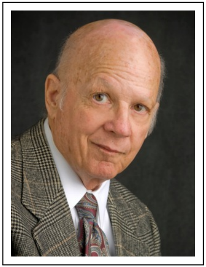
- Belnap argued that instead of using a logic with two values, for 'true' ($\boldsymbol{t}$) and 'false' ($\boldsymbol{f}$), a computer should use a logic with two further values, for 'contradiction' ($\top$) and 'no information' ($\bot$).
- The resulting structure is equipped with two lattice orders, a knowledge or information order and a truth order, and hence is called a bilattice.

$\mathcal{FOUR}$: Belnap's four-valued logic (1977)

"*We want a computer to be able to receive and reason about inconsistent data.*"

Nuel Belnap
University of Pittsburgh since 1961.
Retired in 2011.



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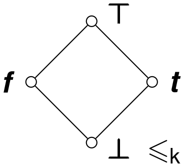
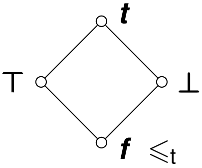
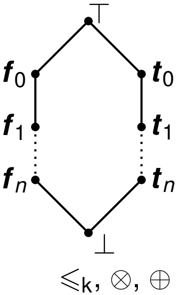
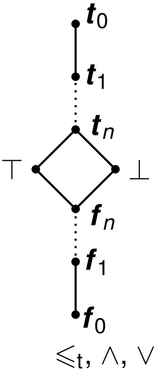




Expanding Belnap's four-element bilattice

We studied a new family of default bilattices. The bilattice $\mathbf{J}_n$ drawn in both its knowledge (left) and truth (right) orders:

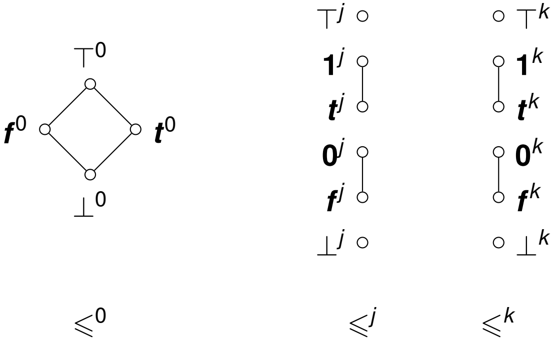




We let $\mathbf{J}_n = \langle J_n; \otimes, \oplus, \wedge, \vee, \neg, C\rangle$ where $C = J_n$ and $\neg \boldsymbol{t}_i = \boldsymbol{f}_i$ and $\neg \boldsymbol{f}_i = \boldsymbol{t}_i$. The negation $\neg$ is an order automorphism w.r.t. the order $\leqslant_{\mathrm{k}}$ and a dual order automorphism w.r.t. the order $\leqslant_{\mathrm{t}}$.

The dualising multi-sorted structure $\mathbb{M}_n$ for $\mathcal{V}_n = \mathrm{Var}(\mathbf{J}_n) = \mathrm{ISP}(\{\mathbf{M}_0, \mathbf{M}_1, \dots, \mathbf{M}_n\})$

- $\leqslant^j$, an order relation on $\mathbf{M}_j$, for $j \in \{0, \dots, n\}$,



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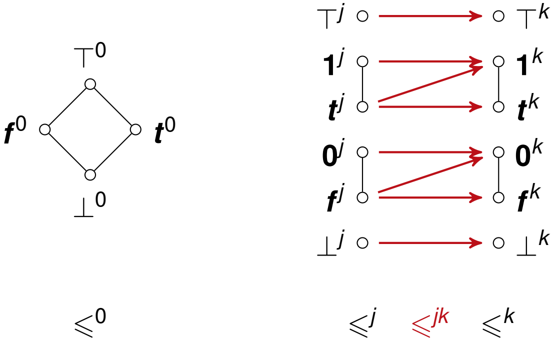
- $\leqslant^j$, an order relation on $\mathbf{M}_j$, for $j \in \{0, \ldots, n\}$,
- $\leqslant^{jk}$, a relation from $\mathbf{M}_j$ to $\mathbf{M}_k$, for $j, k \in \{1, \ldots, n\}$ with $j < k$, can be thought of as the order $\leqslant^j$ 'stretched' from M_j to M_k.



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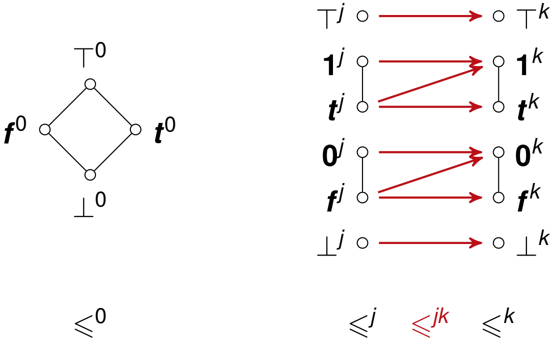

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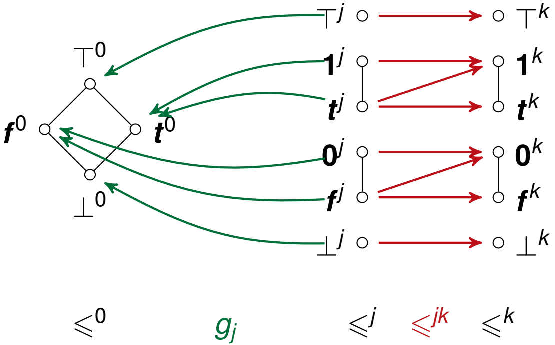
- $g_j \colon \mathbf{M}_j \to \mathbf{M}_0$, the operation that maps $\{\boldsymbol{f}^j, \mathbf{0}^j\}$ to $\boldsymbol{f}^0$ and maps $\{\boldsymbol{t}^j, \mathbf{1}^j\}$ to $\boldsymbol{t}^0$, for $j \in \{1, \dots, n\}$.



A natural duality for the variety $\mathcal{V}_n = \mathrm{Var}(\mathbf{J}_n)$

Theorem

Let $n \in \omega\backslash\{0\}$. Define the multi-sorted alter ego

$$\mathbb{M}_n = \langle M_0 \dot\cup M_1 \dot\cup \cdots \dot\cup M_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathfrak{T}\rangle,$$

where

$$\mathcal{G}_{(n)} = \{\, g_k \mid k \in \{1, \ldots, n\}\,\}, \text{ and}$$

$$\mathcal{S}_{(n)} = \{\, \leqslant^k \mid k \in \{0, \ldots, n\}\,\} \cup \{\, \leqslant^{jk} \mid j, k \in \{1, \ldots, n\} \text{ with } j < k\,\}.$$

The alter ego $\mathbb{M}_n$ yields a duality that is both optimal and full on the variety $\mathcal{V}_n = \mathrm{Var}(\mathbf{J}_n) = \mathrm{ISP}(\{\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_n\})$.

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Our first aim is to give a description of the dual category

$$\mathfrak{X}_n := \mathrm{IS_cP^+}(\mathbb{M}_n).$$

Theorem

$\mathbb{X} = \langle X_0 \dot\cup \cdots \dot\cup X_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T} \rangle$ belongs to $\mathfrak{X}_n := \mathrm{IS_cP^+}(\mathbb{M}_n)$ iff

(A1) $g_k \colon X_k \to X_0$ *is continuous, for all* $k \in [1, n]$;

(A2) $(\forall k \in [1, n]) \quad (\forall x, y \in X_k)\ x \leqslant^k y \implies g_k(x) = g_k(y)$;

(A3) $(\forall j, k \in [1, n]$ *with* $j < k)$

$$(\forall x \in X_j)(\forall y \in X_k)\ x \leqslant^{jk} y \implies g_j(x) = g_k(y);$$

(A4) $(\forall j, k \in [1, n]$ *with* $j < k) \quad (\forall x, y \in X_j)(\forall u, v \in X_k)$

$$x \leqslant^j y \ \&\ y \leqslant^{jk} u \ \&\ u \leqslant^k v \implies x \leqslant^{jk} z;$$

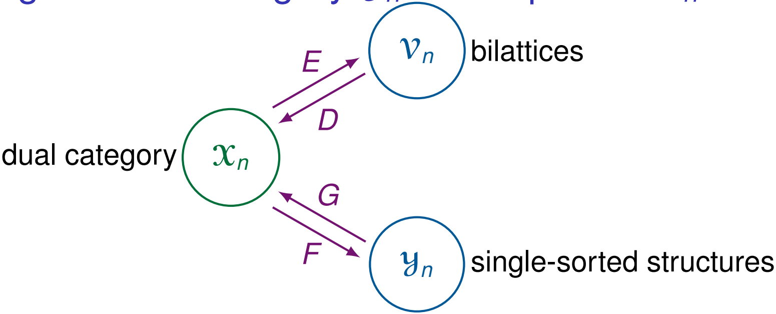
(A5) $(\forall j, k, \ell \in [1, n]$ *with* $j < k < \ell)\ (\forall x \in X_j)(\forall y \in X_k)(\forall z \in X_\ell)$

$$x \leqslant^{jk} y \ \&\ y \leqslant^{k\ell} z \implies x \leqslant^{j\ell} z;$$

(A6) $(\forall k \in [0, n])\ \langle X_k; \leqslant^k, \mathcal{T}_k \rangle$ *is a Priestley space* $(\mathcal{T}_k := \mathcal{T}{\restriction}_{X_k})$;

(A7) $(\forall j, k \in [1, n]$ *with* $j < k)\ (\forall x \in X_j)(\forall y \in X_k)$ *with* $x \not\leqslant^{jk} y$, *there exist* $U_j, U_{j+1}, \ldots, U_k$, *with* U_ℓ *a clopen up-set of* $\langle X_\ell; \leqslant^\ell, \mathcal{T}_\ell \rangle$, *for all* $\ell \in [j, k]$, *such that* $U_j, \ldots, U_k$ *are mutually increasing with* $x \in U_j$ *and* $y \in X_k \backslash U_k$.

A single-sorted category $\mathcal{Y}_n$ isomorphic to $\mathcal{X}_n$



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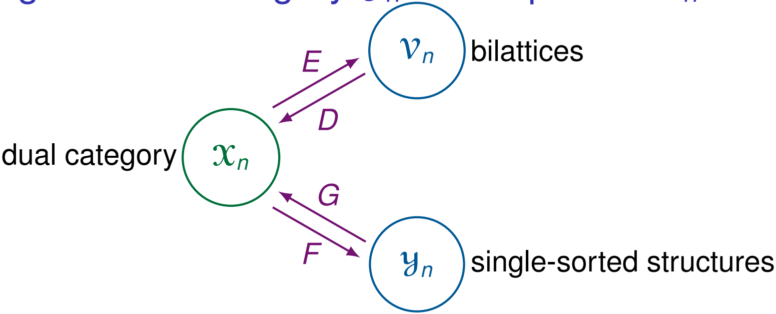

For $\mathbb{X} = \langle X; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T}\rangle \in \mathcal{X}_n$ where $X = X_0 \,\dot\cup \cdots \dot\cup\, X_n$ we define a binary relation $\preccurlyeq$ on X as follows:

$$x \preccurlyeq y :\Leftrightarrow \begin{cases} \big(\exists k \in [0,n]\big)\ x, y \in X_k \text{ and } x \leqslant^k y, \text{ or} \\ \big(\exists j,k \in [1,n] \text{ with } j < k\big)\ \ x \in X_j \ \ \&\ \ y \in X_k \ \ \&\ \ x \leqslant^{jk} y. \end{cases}$$

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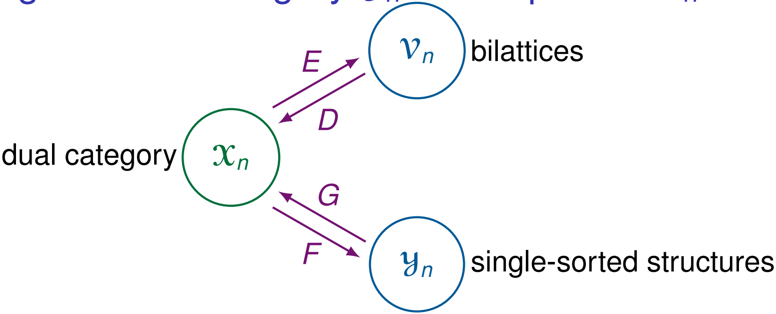

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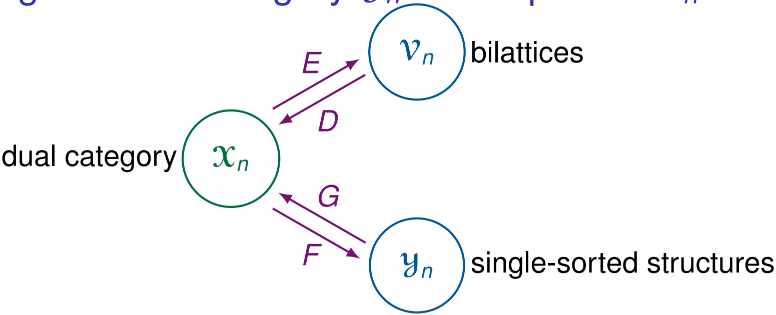

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(a) $\mathbb{X}$ satisfies (A4)–(A6) iff $\preccurlyeq$ is an order on X.

(b) $\mathbb{X}$ satisfies (A4)–(A7) iff $\langle X; \preccurlyeq, \mathcal{T} \rangle$ is a Priestley space.

A single-sorted category $\mathcal{Y}_n$ isomorphic to $\mathcal{X}_n$

Given $n \in \omega \backslash \{0\}$, define an ***n*-ranking** of any Priestley space $\langle X; \preccurlyeq, \mathcal{T} \rangle$ to be a continuous order-preserving map, rnk, from $\langle X; \preccurlyeq, \mathcal{T} \rangle$ to the finite Priestley space $\langle [0, n]; \leqslant, \mathcal{T} \rangle$, where $\leqslant$ is the usual order inherited from $\mathbb{Z}$ and $\mathcal{T}$ is the discrete topology.

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Given $\mathbb{X} \in \mathcal{X}_n$, we can enrich the Priestley space $\langle X; \preccurlyeq, \mathcal{T} \rangle$ with a unary operation g and an n-ranking function defined as follows:

- $g \colon X \to X$ is defined by $g(x) := x$ for all $x \in X_0$ and $g(x) := g_k(x)$, for all $x \in X_k$ and all $k \in [1, n]$,

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We will define a functor $F\colon \mathcal{X}_n \to \mathcal{Y}_n$ by

$$F(\mathbb{X}) := \langle X; \preccurlyeq, g, \mathrm{rnk}, \mathcal{T}\rangle.$$

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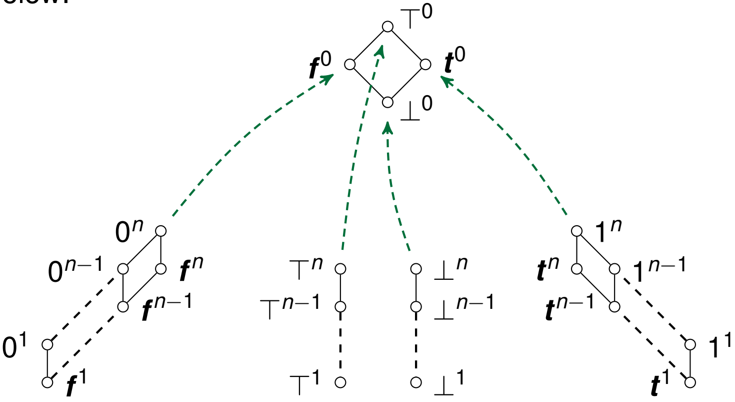
Before abstractly defining $\mathcal{Y}_n$, we present an example of $F(\mathbb{X})$.

A single-sorted category $\mathcal{Y}_n$ isomorphic to $\mathcal{X}_n$

$F(\mathbb{M}_n) = \langle M_0 \,\dot\cup\, M_1 \,\dot\cup \cdots \dot\cup\, M_n; \preccurlyeq, g, \mathrm{rnk}, \mathcal{T}\rangle = F(D(\mathbf{F}_{\mathcal{V}_n}(1)))$ is below:



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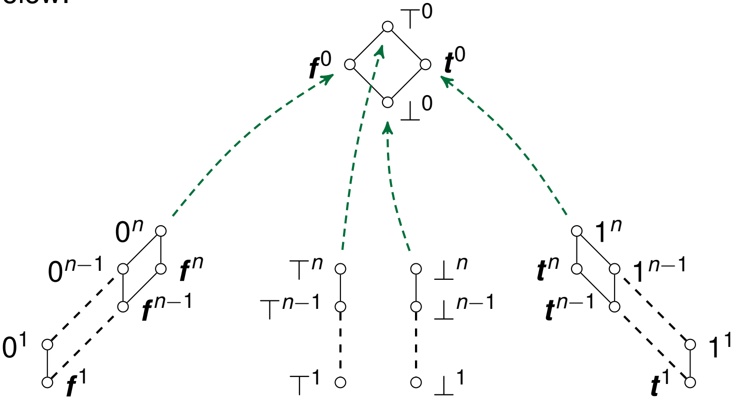
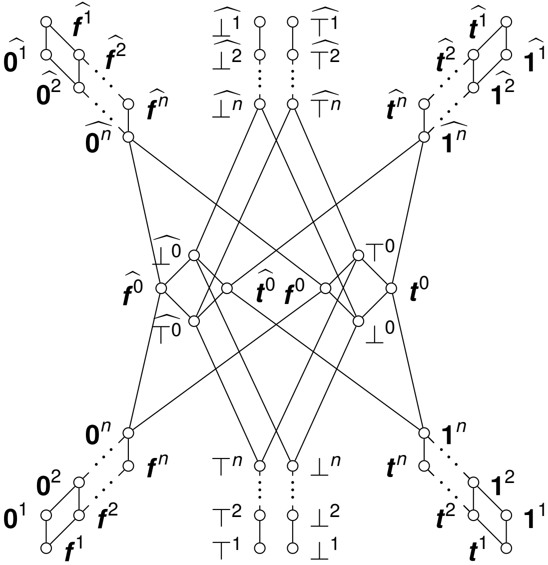


It is interesting to compare this structure with the Priestley dual

$$H(\mathbf{F}_{\mathcal{V}_n}(1)^\flat).$$

The Priestley dual of $\mathbf{F}_{\mathcal{V}_n}(1)^\flat$



A single-sorted category $\mathcal{Y}_n$ isomorphic to $\mathcal{X}_n$

We abstractly define $\mathcal{Y}_n$ to be the category whose objects are topological structures $\langle X; \preccurlyeq, g, \mathrm{rnk}, \mathcal{T}\rangle$ satisfying (B1)–(B6) below and whose morphisms are continuous maps preserving $\preccurlyeq$, g and rnk.

(B1) $\langle X; \preccurlyeq, \mathcal{T}\rangle$ is a Priestley space,
(B2) g is a continuous retraction,
(B3) $x \preccurlyeq y$ implies $g(x) = g(y)$, for all $x, y \in X$,
(B4) $g(X)$ is a union of order components of $\langle X; \preccurlyeq\rangle$,
(B5) $\mathrm{rnk}\colon X \to [0, n]$ is an n-ranking of $\langle X; \preccurlyeq, \mathcal{T}\rangle$,
(B6) $g(X) = \{\, x \in X \mid \mathrm{rnk}(x) = 0 \,\}$.

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Note: The topological structures in $\mathcal{Y}_n$ can be made first order by removing the ‘operation’ rnk, adding $n + 1$ topologically closed unary relations $X_0, \dots, X_n$, and replacing the assumption that rnk is order-preserving by the assumption that the X_k partition X plus the following axioms:

$$(\forall k \in [0, n])\ \ (\forall x, y \in X)\ x \in X_k \ \ \&\ \ x \preccurlyeq y \implies y \in X_k \text{ or } y \in X_{k+1} \text{ or } \cdots \text{ or } y \in X_n.$$

A single-sorted category $\mathcal{Y}_n$ isomorphic to $\mathcal{X}_n$

Let $\mathbb{Y} = \langle Y; \preccurlyeq, g, \mathrm{rnk}, \mathcal{T} \rangle$ be an object in $\mathcal{Y}_n$. We define

$$G(\mathbb{Y}) := \langle X_0 \dot{\cup} \cdots \dot{\cup} X_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T} \rangle$$

in the signature of $\mathbb{M}_n$ as follows:

- $X_k := \{ x \in Y \mid \mathrm{rnk}(x) = k \}$, for all $k \in [0, n]$,
- $g_k \colon X_k \to X_0$ is given by $g_k := g{\restriction}_{X_k}$, for all $k \in [1, n]$,
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A single-sorted category $\mathcal{Y}_n$ isomorphic to $\mathcal{X}_n$

Let $\mathbb{Y} = \langle Y; \preccurlyeq, g, \mathrm{rnk}, \mathcal{T}\rangle$ be an object in $\mathcal{Y}_n$. We define

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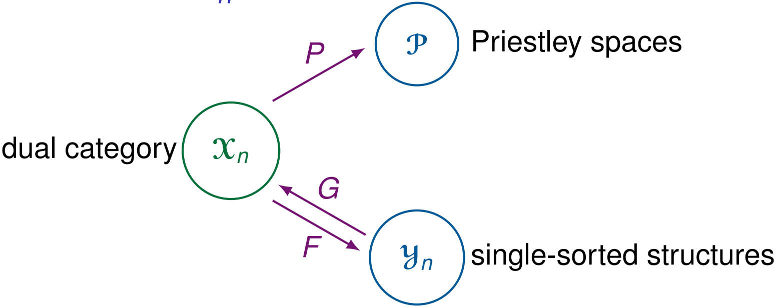
Theorem

$F \colon \mathcal{X}_n \to \mathcal{Y}_n$ and $G \colon \mathcal{Y}_n \to \mathcal{X}_n$ are well-defined, mutually inverse category isomorphisms. In particular,

- $G(F(\mathbb{X})) = \mathbb{X}$, *for all* $\mathbb{X} \in \mathcal{X}_n$, *and*
- $F(G(\mathbb{Y})) = \mathbb{Y}$, *for all* $\mathbb{Y} \in \mathcal{Y}_n$.

The functor $\mathrm{P}\colon \mathcal{X}_n \to \mathcal{P}$



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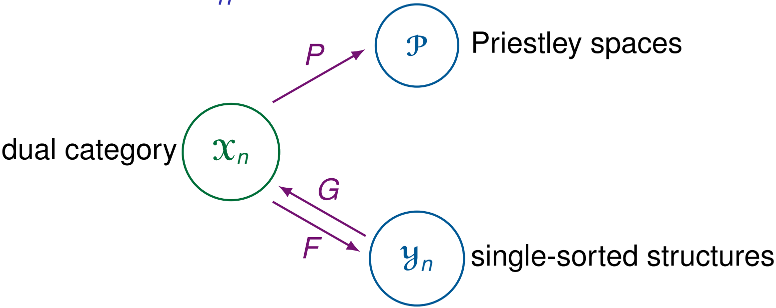

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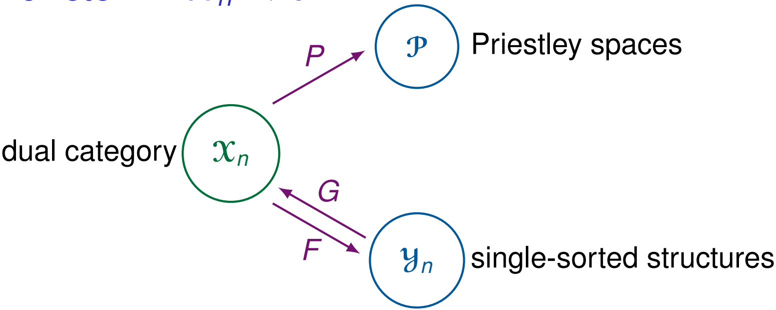

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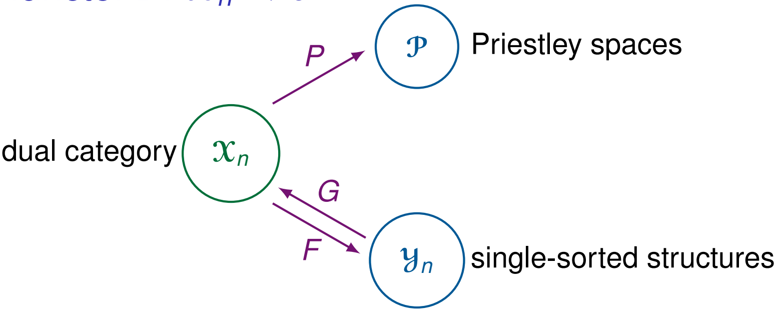

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Definition

Let $\mathbb{X} = \langle X_0 \,\dot\cup \cdots \dot\cup\, X_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T}\rangle$ be an object in $\mathcal{X}_n$, define $X := X_0 \,\dot\cup \cdots \dot\cup\, X_n$ and let $\mathrm{F}(\mathbb{X}) = \langle X; g, \leqslant, \mathrm{rnk}, \mathcal{T}\rangle$ be the corresponding object in $\mathcal{Y}_n$. We define $\mathrm{P}(\mathbb{X}) := \langle X \,\dot\cup\, \widehat{X}; \preccurlyeq, \mathcal{T}\rangle$ where $\widehat{X} := \{\widehat{x} \mid x \in X\}$, $\mathcal{T}$ is the disjoint union topology, and the order $\preccurlyeq$ is given on $X \,\dot\cup\, \widehat{X}$ as on the next page.

The order in the constructed Priestley spaces

Definition

- for $x, y \in X$: $x \preccurlyeq y \iff x \leqslant y$,
- for $\widehat{x}, \widehat{y} \in \widehat{X}$: $\widehat{x} \preccurlyeq \widehat{y} \iff x \geqslant y$,
- for $x \in X \backslash X_0$ and $y \in X_0$: $x \preccurlyeq y \iff g(x) \leqslant y$,
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- for $x \in X \backslash X_0$ and $\widehat{y} \in \widehat{X} \backslash \widehat{X}_0$:
  $x \preccurlyeq \widehat{y} \iff g(x) \leqslant g(y)$ or $g(x) \geqslant g(y)$.

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The last case of the definition of $\preccurlyeq$ can be thought of as obtaining $x \preccurlyeq \widehat{y}$ by passing through X_0 (via $g(x) \leqslant g(y)$) or through $\widehat{X}_0$ (via $g(x) \geqslant g(y)$).

Example of $\mathrm{P}(\mathbb{X})$

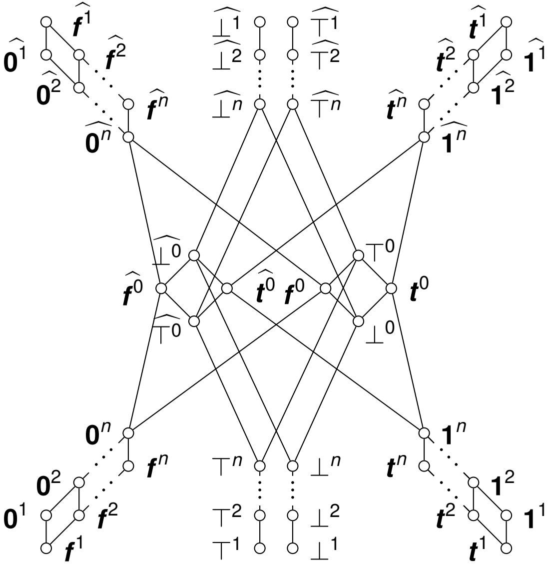

Figure: The ordered set $\mathrm{P}(\mathbb{M}_n)$ - the Priestley dual of $\mathbf{F}_{\mathcal{V}_n}(1)^\flat$.

Theorem and its application

Theorem

Fix $n \in \omega \backslash \{0\}$. Then $\mathrm{P}\colon \mathfrak{X}_n \to \mathcal{P}$ is a well-defined functor. For each prioritised default bilattice $\mathbf{A} \in \mathcal{V}_n$, let $\mathbf{A}^\flat = \langle A; \wedge, \vee, \boldsymbol{f}_0, \boldsymbol{t}_0 \rangle$ be its bounded-distributive lattice truth reduct. Then its Priestley dual $H(\mathbf{A}^\flat)$ is isomorphic to $\mathrm{P}(\mathrm{D}(\mathbf{A}))$.

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  - the top part **T**, also isomorphic to $(\mathbf{2} \times \mathbf{n}) \dot\cup \mathbf{n} \dot\cup \mathbf{n} \dot\cup (\mathbf{2} \times \mathbf{n})$.

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    - the top part **T**, also isomorphic to $(\mathbf{2} \times \mathbf{n}) \dot\cup \mathbf{n} \dot\cup \mathbf{n} \dot\cup (\mathbf{2} \times \mathbf{n})$.
- We counted the down-sets of $\mathrm{P}(\mathbb{M}_n)$ by first counting the number that do not intersect the top **T** and then counting the number that do.

The calculation

Claim

The number of down-sets of $\mathrm{P}(\mathbb{M}_n)$ *that do not intersect* **T** *is*

$$f(n) = \frac{1}{4}\left(n^6 + 10n^5 + 41n^4 + 96n^3 + 148n^2 + 148n + 144\right).$$

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Let U be a down-set of $\mathrm{P}(\mathbb{M}_n)$ that does not intersect **T.** The intersection $U \cap C$ is one of the 36 down-sets of **C.** The number of such U for a given intersection $U \cap C$ is given in the table:

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$U \cap C$	# of such U
$\varnothing$	$\frac{1}{4}(n+1)^4(n+2)^2$
$\{\bot\}$ or $\{\widehat{\top}\}$	$\frac{1}{4}(n+1)^3(n+2)^2$
$\{\bot, \boldsymbol{f}\}$ or $\{\bot, \boldsymbol{t}\}$ or $\{\widehat{\top}, \widehat{\boldsymbol{f}}\}$ or $\{\widehat{\top}, \widehat{\boldsymbol{t}}\}$	$\frac{1}{2}(n+1)^2(n+2)$
$\{\bot, \boldsymbol{f}, \boldsymbol{t}\}$ or $\{\widehat{\top}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{t}}\}$	$n+1$
$\{\bot, \widehat{\top}\}$	$\frac{1}{4}(n+1)^2(n+2)^2$
$\{\bot, \widehat{\top}, \widehat{\boldsymbol{f}}\}$ or $\{\bot, \widehat{\top}, \widehat{\boldsymbol{t}}\}$ or $\{\bot, \boldsymbol{f}, \widehat{\top}\}$ or $\{\bot, \boldsymbol{t}, \widehat{\top}\}$ or $\{\bot, \boldsymbol{f}, \widehat{\top}, \widehat{\boldsymbol{f}}\}$ or $\{\bot, \boldsymbol{t}, \widehat{\top}, \widehat{\boldsymbol{t}}\}$	$\frac{1}{2}(n+1)(n+2)$
each of the remaining 20 possibilities	1

The calculation - continuation

Claim

The number of down-sets of $\mathrm{P}(\mathbb{M}_n)$ *that intersect the top* **T** *is*

$$g(n) = \frac{1}{4}\left(n^6 + 10n^5 + 43n^4 + 108n^3 + 166n^2 + 148n\right).$$

The calculation - continuation

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The number of down-sets of $\mathrm{P}(\mathbb{M}_n)$ *that intersect the top* **T** *is*

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A down-set U of $\mathrm{P}(\mathbb{M}_n)$ that intersects **T** does intersect the set $\min(\mathbf{T}) = \{\widehat{\mathbf{0}^n}, \widehat{\bot^n}, \widehat{\top^n}, \widehat{\mathbf{1}^n}\}$ of minimal elements of **T** in one of the 15 non-empty subsets of $\min(\mathbf{T})$ given in the table below:

$U \cap \min(\mathbf{T})$	# of such U
$\{\widehat{\mathbf{0}}\}$ or $\{\widehat{\mathbf{1}}\}$	$\left(\frac{1}{2}(n+1)(n+2)-1\right)\left(\frac{1}{2}(n+1)(n+2)+8\right)$
$\{\widehat{\bot}\}$ or $\{\widehat{\top}\}$	$5n$
$\{\widehat{\mathbf{0}}, \widehat{\mathbf{1}}\}$	$4\left(\frac{1}{2}(n+1)(n+2)-1\right)^2$
$\{\widehat{\mathbf{0}}, \widehat{\top}\}$ or $\{\widehat{\mathbf{0}}, \widehat{\bot}\}$ or $\{\widehat{\top}, \widehat{\mathbf{1}}\}$ or $\{\widehat{\bot}, \widehat{\mathbf{1}}\}$	$3n\left(\frac{1}{2}(n+1)(n+2)-1\right)$
$\{\widehat{\top}, \widehat{\bot}\}$	n^2
$\{\widehat{\bot}, \widehat{\top}, \widehat{\mathbf{1}}\}$ or $\{\widehat{\mathbf{0}}, \widehat{\top}, \widehat{\bot}\}$	$n^2\left(\frac{1}{2}(n+1)(n+2)-1\right)$
$\{\widehat{\mathbf{0}}, \widehat{\top}, \widehat{\mathbf{1}}\}$ or $\{\widehat{\mathbf{0}}, \widehat{\bot}, \widehat{\mathbf{1}}\}$	$2n\left(\frac{1}{2}(n+1)(n+2)-1\right)^2$
$\{\widehat{\mathbf{0}}, \widehat{\bot}, \widehat{\top}, \widehat{\mathbf{1}}\}$	$n^2\left(\frac{1}{2}(n+1)(n+2)-1\right)^2$

The final result

Theorem

Let $n \in \omega \backslash \{0\}$. Then the cardinality of the 1-generated free algebra in the variety $\mathcal{V}_n = \mathrm{Var}(\mathbf{J}_n)$ is

$$|F_{\mathcal{V}_n}(1)| = \frac{1}{2}\big(n^6 + 10n^5 + 42n^4 + 102n^3 + 157n^2 + 148n + 72\big).$$

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References

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Thank you for your attention!