Congruence lattices of Abelian l-groups

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Abelian l-groups

Algebras $(G, +, 0, -, \vee, \wedge)$ such that

(i) $(G, +, 0, -)$ is an Abelian group;

(ii) $(G, \vee, \wedge)$ is a lattice;

(iii) $x \leq y$ implies $x + z \leq y + z$ for every $x, y, z \in G$.

Abelian l-groups form a variety generated by $\mathbb{Z}$.

Congruences and ideals

Congruences on Abelian l-groups correspond to l-ideals (=convex l-subgroups), these form a distributive algebraic lattice $\operatorname{Id} G$.
Compact congruences correspond to compact (finitely generated) l-ideals. They are principal and have the form

$$\langle a \rangle = \{x \in G \mid (-na) \leq x \leq na \text{ for some } n \in \omega\},$$

for $a \geq 0$.
Compact l-ideals form a sublattice $\operatorname{Id}_c G$ of $\operatorname{Id} G$. The lattice $\operatorname{Id} G$ is determined by $\operatorname{Id}_c G$ uniquely.

Problem

Problem. Which distributive lattices are isomorphic to $\mathrm{Id}_c\, G$ for some Abelian l-group G?

Equivalent form:
Characterize spectral spaces of Abelian l-groups. (The set of prime ideals endowed with the hull-kernel topology.)

Complete normality

Known for > 40 years:

Theorem

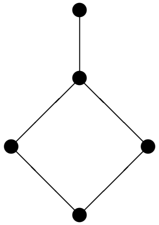
Every lattice $\mathrm{Id}_c\, G$ is completely normal, i.e. satisfies

$$(\forall a, b)(\exists x, y)(a \vee b = a \vee y = x \vee b \text{ and } x \wedge y = 0).$$

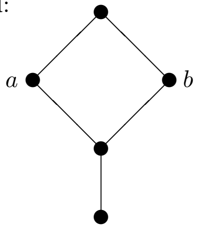
Intuitively: $x = a \setminus b$, $y = b \setminus a$.
Equivalently: the ordered set of all prime ideals of the lattice $\mathrm{Id}_c\, G$ is a root system. (poset in which $\uparrow x$ is a chain for every x)

Good and bad

good:



bad:



Finite case

Theorem

Every finite completely normal distributive lattice is isomorphic to $\mathrm{Id}_c G$ *for some Abelian* l*-group* G*.*

Construction (Conrad 1965): Let P be a finite root system. Let $G = \mathbb{Z}^P$ be the set of all functions $P \to \mathbb{Z}$ with poinwise addition and the order given by

$$f > g \quad \text{iff} \quad f(x) > g(x) \text{ for every } x \in M,$$

where $M = \max\{x \in P \mid f(x) \neq g(x)\}$.

Special infinite case 1

Let L be a Boolean lattice, $L \subseteq \mathcal{P}(X)$.
Let G be the subalgebra of $\mathbb{Z}^X$ generated by all characteristic functions χ_A for $A \in L$. Then L is isomorphic to $\mathrm{Id}_c\, G$.

Special infinite case 2

Let L be a chain with the least element.
Let G be the set of all functions $L \to \mathbb{Z}$ with finite support. The addition is pointwise, the order lexicographic, which means

$$f < g \quad \text{iff} \quad f(t) < g(t),$$

where $t = \max\{x \in L \mid f(x) \neq g(x)\}$.
Then G is an Abelian l-group and L is isomorphic to $\operatorname{Id}_c G$.

Complete normality sufficient?

No.
(Example by Delzell and Madden 1994)

A new necessary condition found by Cignoli, Gluschankof and Lucas (1999),
and also by Iberkleid, Martinez and McGovern (2011)

Countably based differences

For $a, b \in L$ define

$$a \ominus b = \{x \in L \mid a \leq b \vee x\}.$$

We say that the lattice L *has countably based differences*, if the set $a \ominus b$ has a countable coinitial subset for every $a, b \in L$.

Theorem

For every Abelian l-group G, the lattice $\mathrm{Id}_c G$ has countably based differences.

Questions

(1) Is every distributive completely normal lattice with countably based differences representable as $\mathrm{Id}_c\, G$?

(2) Is every *countable* distributive completely normal lattice representable as $\mathrm{Id}_c\, G$?

Both questions were recently solved by F. Wehrung.

Cevian operations

A binary operation $\setminus$ on a distributive lattice L with 0 is called Cevian, if

(i) $y \vee (x \setminus y) \geq x$ for every $x, y \in L$;

(ii) $(x \setminus y) \wedge (y \setminus x) = 0$ for every $x, y \in L$;

(iii) $x \setminus z \leq (x \setminus y) \vee (y \setminus z)$ for every $x, y, z \in L$.

The lattice L is Cevian, if it has a Cevian operation. Clearly, every Cevian lattice is completely normal.

Wehrung's solutions

Theorem

- *For every Abelian l-group G, the lattice $\mathrm{Id}_c\, G$ is Cevian.*
- *There exists a non-Cevian distributive lattice L of cardinality $\aleph_2$, which is completely normal and has countably based differences.*

Theorem

Every countable completely normal distributive lattice is representable as $\mathrm{Id}_c\, G$.

Remark: Delzell-Madden example has cardinality $\aleph_1$.

New questions

(1) Is every Cevian distributive lattice with countably based differences representable as $\mathrm{Id}_c\, G$?

(2) Is every completely normal distributive lattice with 0 of cardinality $\leq \aleph_1$ Cevian?

(3) Is every completely normal distributive lattice with 0 of cardinality $\aleph_1$ having countably based differences representable as $\mathrm{Id}_c\, G$?

(1) and (2) recently solved by M. Ploščica.

Construction

Let C be a bounded chain. Let L_C be the sublattice of $C \times \mathcal{P}(\omega)$ defined by

$$(\alpha, A) \in L_C \quad \text{iff} \quad (\alpha = 0 \text{ and } A \text{ is finite}) \text{ or } (A \text{ is cofinite}).$$

Easy to prove:

Theorem

For every chain C, the lattice L_C is Cevian and has countably based differences.

Solution to (1)

Theorem

If L_C is isomorphic to $\mathrm{Id}_c\, G$, then $|C| \leq 2^\omega$

Proof: L_C contains countably many coatoms c_i such that $\bigwedge c_i = 0$. If L_C is isomorphic to $\mathrm{Id}_c\, G$, then G contains countably many maximal l-ideals with intersection equal to $\{0\}$. Hence, G is a subdirect product of countably many simple algebras. However, simple l-groups embeds in reals, so $|G| \leq (2^\omega)^\omega = 2^\omega$, and then $|C| \leq |L_C| \leq |G| \leq 2^\omega$.

So, if we choose C with $|C| > 2^\omega$, then L_C is a nonrepresentable Cevian lattice with countably based differences. Under Continuum Hypothesis we can have $|L_C| = \aleph_2$.

Which L_C are representable?

On the set of all functions $\omega \to \omega^+$ we define

$$f \sqsubset g \quad \text{iff} \quad (\text{the set } \{x \mid nf(x) > g(x)\} \text{ is finite for every } n \in \omega)$$

Equivalently: $\lim g(x)/f(x) = \infty$.

Theorem

Suppose that there are functions h_α, $\alpha \in C$, such that $h_\alpha \sqsubset h_\beta$ whenever $\alpha < \beta$. Then L_C is representable.

I can find such h_α whenever $|C| \leq \aleph_1$.
Conjecture: The above Theorem is an equivalence (true under CH).

Properties of $\sqsubset$

- If $f \sqsubset g$, then $f \sqsubset h \sqsubset g$ for some h.
- If $f_0 \sqsubset f_1 \sqsubset f_2 \sqsubset \cdots \sqsubset g$, then there exists h such that $f_i \sqsubset h \sqsubset g$ for every i.
- If $g \sqsubset \cdots \sqsubset f_2 \sqsubset f_1 \sqsubset f_0$, then there exists h such that $g \sqsubset h \sqsubset f_i$ for every i.
- If $g_0 \sqsubset g_1 \sqsubset g_2 \sqsubset \cdots \sqsubset f_2 \sqsubset f_1 \sqsubset f_0$, then there exists h such that $g_i \sqsubset h \sqsubset f_j$ for every i, j.

Solution to (2)

Theorem

Every completely normal distributive lattice of cardinality at most $\aleph_1$ is Cevian.

This leaves (3) open. So, once again:

Problem: Is every completely normal distributive lattice of cardinality $\aleph_1$ with countably based differences isomorphic to $\mathrm{Id}_c\, G$ for some Abelian l-group G?

A possible strategy: to generalize Wehrung's proof for countable extensions of lattices instead of countable lattices.

Thanks

Thank you for attention.

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