Sierpinski object for composite affine spaces

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Outline

Sierpinski space and object

- The notion of *Sierpinski space* $\mathcal{S} = (\{0,1\}, \{\varnothing, \{1\}, \{0,1\}\})$ plays an important role in general topology, e.g., a topological space is T_0 iff it can be embedded into some power of $\mathcal{S}$.
- E. G. Manes introduced the concept of *Sierpinski object* in categories of structured sets and structure-preserving maps, and characterized the category of topological spaces among such categories in terms of the Sierpinski object.
- There exists a characterization of the category of fuzzy topological spaces in terms of the Sierpinski object of E. G. Manes.

Affine topology

- There exists an affine approach to lattice-valued topology, inspired by the notion of *affine set* of Y. Diers.
- While lattice-valued topology replaces the two-element Boolean algebra with some lattice-theoretic structure (e.g., a quantale), affine topology uses an algebra from an arbitrary variety.
- *Composite affine spaces* provide an analogue of bitopological spaces of J. C. Kelly (sets with two topologies), i.e., these are the spaces, which have a set-indexed family of affine topologies.

Composite affine Sierpinski space

- Recently, R. Noor *et al.* gave a characterization of Sierpinski object in the category of affine bitopological spaces.
- We describe Sierpinski object in the category of composite affine spaces: we construct a functor from the category of affine spaces to that of composite affine spaces, and show a simple condition, under which this functor preserves Sierpinski objects.
- We therefore provide a convenient method to obtain a composite affine Sierpinski space, given an affine Sierpinski space, getting, in particular, the Sierpinski object of R. Noor *et al.*

Ω-algebras and Ω-homomorphisms

Definition 1

Let $\Omega = (n_\lambda)_{\lambda\in\Lambda}$ be a family of cardinal numbers, which is indexed by a (possibly proper or empty) class Λ.

- An *Ω-algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda\in\Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (*n_λ-ary primitive operations* on A).
- An *Ω-homomorphism* $(A_1, (\omega_\lambda^{A_1})_{\lambda\in\Lambda}) \xrightarrow{\varphi} (A_2, (\omega_\lambda^{A_2})_{\lambda\in\Lambda})$ is a map $A_1 \xrightarrow{\varphi} A_2$ such that $\varphi \circ \omega_\lambda^{A_1} = \omega_\lambda^{A_2} \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$.
- $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms.

Forgetful functors of concrete categories will be denoted $|-|$.

Varieties and algebras

Definition 2

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of Ω-algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients, and whose objects (resp. morphisms) are called *algebras* (resp. *homomorphisms*).

Example 3

1. $\mathbf{CSLat}(\bigvee)$ is the variety of *$\bigvee$-semilattices*.
2. **(U)SQuant** is the variety of (*unital*) *semi-quantales*.
3. **Frm** is the variety of *frames*.
4. **CBAlg** is the variety of *complete Boolean algebras*.
5. **CL** is the variety of *closure lattices*.

Affine spaces

From now on, $\mathbf{A}$ stands for an arbitrary variety of algebras.

Definition 4

Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$, $\mathbf{Af\,Spc}(T)$ denotes the concrete category over $\mathbf{X}$, whose

objects (*T-affine spaces* or *T-spaces*) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ is a subalgebra of TX;

morphisms (*T-affine morphisms* or *T-morphisms*) $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are $\mathbf{X}$-morphisms $X_1 \xrightarrow{f} X_2$ with the property that $(Tf)^{op}(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$.

Examples of the functor T

Proposition 5

Every subcategory $\mathbf{S}$ *of* $\mathbf{A}^{op}$ *provides a functor* $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{P}_{\mathbf{S}}} \mathbf{A}^{op}$, *which is given by* $\mathcal{P}_{\mathbf{S}}((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)) = A_1^{X_1} \xrightarrow{\mathcal{P}_{\mathbf{S}}(f,\varphi)} A_2^{X_2}$, $(\mathcal{P}_{\mathbf{S}}(f,\varphi))^{op}(\alpha) = \varphi^{op} \circ \alpha \circ f$.

Example 6

1. A subcategory $\mathbf{S} = \{A \xrightarrow{1_A} A\}$ provides a functor $\mathbf{Set} \xrightarrow{\mathcal{P}_A} \mathbf{A}^{op}$, $\mathcal{P}_A(X_1 \xrightarrow{f} X_2) = A^{X_1} \xrightarrow{\mathcal{P}_A f} A^{X_2}$, where $(\mathcal{P}_A f)^{op}(\alpha) = \alpha \circ f$.
2. For $\mathbf{A} = \mathbf{CBAlg}$ and $\mathbf{S} = \{2 \xrightarrow{1_2} 2\}$, one gets the contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$, given on a map $X_1 \xrightarrow{f} X_2$ by $\mathcal{P}X_2 \xrightarrow{(\mathcal{P}f)^{op}} \mathcal{P}X_1$ where $(\mathcal{P}f)^{op}(S) = \{x \in X_1 \mid f(x) \in S\}$.

Examples of affine spaces

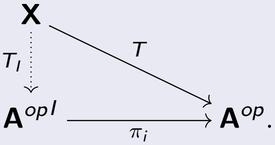
Example 7

1. If $\mathbf{A} = \mathbf{Frm}$, then $\mathbf{Af\,Spc}(\mathcal{P}_2)$ is the category **Top** of topological spaces.
2. $\mathbf{Af\,Spc}(\mathcal{P}_A)$ is the category $\mathbf{Af\,Set}(A)$ of affine sets of Y. Diers.
3. If $\mathbf{A} = \mathbf{USQuant}$ or $\mathbf{A} = \mathbf{Frm}$, then $\mathbf{Af\,Spc}(\mathcal{P}_{\mathbf{S}})$ is the category **S-Top** of variable-basis lattice-valued topological spaces of S. E. Rodabaugh.
4. If $\mathbf{A} = \mathbf{CL}$, then $\mathbf{Af\,Spc}(\mathcal{P}_2)$ is the category **Cls** of closure spaces of D. Aerts *et al.*

Definition of the functor T_I

Definition 8

Let $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ be a functor, and let I be a *non-empty* set. Then $\mathbf{X} \xrightarrow{T_I} \mathbf{A}^{op\,I}$ stands for a functor, which is given by the commutative (for every $i \in I$) triangle



Composite affine spaces

Definition 9

Given a functor $\mathbf{X} \xrightarrow{T_I} \mathbf{A}^{op\,I}$, $\mathbf{CAf\,Spc}(T_I)$ denotes the concrete category over $\mathbf{X}$, whose

objects (*composite T-affine spaces* or *composite T-spaces*) are pairs $(X, (\tau_i)_{i \in I})$, where X is an $\mathbf{X}$-object and τ_i is a subalgebra of TX for every $i \in I$;

morphisms (*composite T-affine morphisms* or *composite T-morphisms*) $(X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ such that for every $i \in I$, $(Tf)^{op}(\alpha) \in \tau_i$ for every $\alpha \in \sigma_i$.

Examples of composite affine spaces

Example 10

1. If $\mathbf{A} = \mathbf{Frm}$ and $I = \{1, 2\}$, then $\mathbf{CAf\,Spc}(\mathcal{P}_{2I})$ is the category **BiTop** of bitopological spaces of J. C. Kelly.
2. If $\mathbf{A} = \mathbf{USQuant}$ and $I = \{1, 2\}$, then $\mathbf{CAf\,Spc}(\mathcal{P}_{AI})$ is the category A-**BiTop** of lattice-valued bitopological spaces of S. E. Rodabaugh.
3. If $I = \{1, 2\}$, then $\mathbf{CAf\,Spc}(\mathcal{P}_{AI})$ is the category of A-bitopological spaces of R. Noor *et al.*

Initial sources and topological categories

Definition 11

Given a concrete category **C** over **X**, a source $(C \xrightarrow{f_i} C_i)_{i \in I}$ in **C** is *initial* provided that an **X**-morphism $|B| \xrightarrow{f} |C|$ is a **C**-morphism whenever each composite $|B| \xrightarrow{f_i \circ f} |C_i|$ is a **C**-morphism.

Theorem 12

The concrete category $(\mathbf{CAf\,Spc}(T_I), |-|)$ *is topological over* **X**.

Corollary 13

The category $\mathbf{CAf\,Spc}(T_I)$ *has (co)products provided that* **X** *does.*

From affine spaces to composite affine spaces

Proposition 14

There exists a functor $\mathbf{Af\,Spc}(T) \xrightarrow{F} \mathbf{CAf\,Spc}(T_I)$, $F((X,\tau) \xrightarrow{f} (Y,\sigma)) = (X^I,(\tau_i)_{i\in I}) \xrightarrow{f^I} (Y^I,(\sigma_i)_{i\in I})$, *where* $(X^I,\tau_i) \xrightarrow{\pi_i} (X,\tau)$ *is an initial morphism in* $\mathbf{Af\,Spc}(T)$ *for every* $i \in I$.

Proposition 15

$\mathbf{Af\,Spc}(T) \xrightarrow{F} \mathbf{CAf\,Spc}(T_I)$ *preserves initial sources.*

Proposition 16

$\mathbf{Af\,Spc}(\mathcal{P}_A) \xrightarrow{F} \mathbf{CAf\,Spc}(\mathcal{P}_{A_I})$ *is a non-concrete embedding.*

From composite affine spaces to affine spaces

From now on, assume that the category **X** has (co)products.

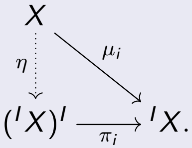
Proposition 17

There exists a functor $\mathbf{CAf\,Spc}(T_I) \xrightarrow{G} \mathbf{Af\,Spc}(T)$, *which is given by* $G((X,(\tau_i)_{i\in I}) \xrightarrow{f} (Y,(\sigma_i)_{i\in I})) = \coprod_{i\in I}(X,\tau_i) \xrightarrow{{}^I f} \coprod_{i\in I}(Y,\sigma_i)$.

A specific morphism η

Remark 18

There exists an $\mathbf{X}$*-morphism* $X \xrightarrow{\eta} ({}^I X)^I$*, which is given by commutativity (for every* $i \in I$*) of the following diagram:*



Proposition 19

$(X, (\tau_i)_{i \in I}) \xrightarrow{\eta} FG(X, (\tau_i)_{i \in I})$ *is a* $\mathbf{CAf\,Spc}(T_I)$*-morphism.*

Properties of the morphism η

Assumption 20

Given an $\mathbf{X}$*-object* X *and a set* I*, form a coproduct* $(X \xrightarrow{\mu_i} {}^I X)_{i \in I}$ *in* $\mathbf{X}$ *and get a source* $(T({}^I X) \xrightarrow{(T\mu_i)^{op}} TX)_{i \in I}$ *in* $\mathbf{A}$*. Suppose a family* $(\tau_i)_{i \in I}$ *of subalgebras of* TX *is given. Assume the following: for every* $i_0 \in I$ *and every* $\alpha \in \tau_{i_0}$*, there is* $\gamma \in T({}^I X)$ *such that* $(T\mu_{i_0})^{op}(\gamma) = \alpha$ *and, for every* $i \in I$ *with* $i \neq i_0$*,* $(T\mu_i)^{op}(\gamma) \in \tau_i$*.*

Proposition 21

$\mathbf{Set} \xrightarrow{\mathcal{P}_A} \mathbf{A}^{op}$ *satisfies Assumption 20 if* $\mathbf{A}$ *has nullary operations.*

Example 22

The powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$ satisfies Assumption 20.

Relationships between the functors F and G

Theorem 23

There is an adjoint situation $G \dashv F : \mathbf{Af\,Spc}(T) \to \mathbf{CAf\,Spc}(T_I)$.

Proposition 24

If Assumption 20 holds, then the category $\mathbf{Af\,Spc}(\mathcal{P}_A)$ is (non-concretely) isomorphic to an (initially) reflective subcategory of the category $\mathbf{CAf\,Spc}(\mathcal{P}_{AI})$, in which "initially" means that every reflection arrow is an initial morphism.

Affine Sierpinski space

Definition 25

In a concrete category **C**, a **C**-object S is a *Sierpinski object* provided that for every **C**-object C, the hom-set $\mathbf{C}(C, S)$ is an initial source.

Example 26

1. In the category $\mathbf{Af\,Spc}(\mathcal{P}_A)$, Sierpinski object (or *Sierpinski (A-)affine space*) is the pair $\mathcal{S} = (|A|, \langle 1_A \rangle)$, where $\langle 1_A \rangle$ is the subalgebra of $A^{|A|}$, which is generated by the identity map 1_A.
2. In the category **Top**, Sierpinski object is the standard Sierpinski space $\mathcal{S} = (\{0, 1\}, \{\varnothing, \{1\}, \{0, 1\}\})$.

Composite affine Sierpinski space

Theorem 27

Suppose Assumption 20 holds. If $\mathcal{S}$ is a Sierpinski object in $\mathbf{Af\,Spc}(T)$, then $F\mathcal{S}$ is a Sierpinski object in $\mathbf{CAf\,Spc}(T_I)$.

To construct a Sierpinski object in the category $\mathbf{CAf\,Spc}(T_I)$, apply the functor F to a Sierpinski object in the category $\mathbf{Af\,Spc}(T)$, which is usually well-known in each concrete case.

An example of composite affine Sierpinski space

In the following, suppose that Assumption 20 holds.

Example 28

- $\mathcal{S} = (|A|, \langle 1_A \rangle)$ is a Sierpinski object in $\mathbf{Af\,Spc}(\mathcal{P}_A)$.
- $F\mathcal{S} = (A^I, (\tau_i)_{i \in I})$ is a Sierpinski object in $\mathbf{CAf\,Spc}(\mathcal{P}_{A^I})$.
- For every $i \in I$, $(A^I, \tau_i) \xrightarrow{\pi_i} (A, \langle 1_A \rangle)$ is an initial morphism in $\mathbf{Af\,Spc}(\mathcal{P}_A)$, and, therefore, $\tau_i = (\mathcal{P}_A \pi_i)^{op}(\langle 1_A \rangle) = \langle ((\mathcal{P}_A \pi_i)^{op} 1_A) \rangle = \langle 1_A \circ \pi_i \rangle = \langle \pi_i \rangle$.
- A Sierpinski object in the category $\mathbf{CAf\,Spc}(\mathcal{P}_{A^I})$ is therefore $F\mathcal{S} = (A^I, (\langle \pi_i \rangle)_{i \in I})$, which is the form of Sierpinski object considered for affine bitopological spaces by R. Noor *et al.*
- In the category **BiTop**, Sierpinski object is the space $\mathcal{S}_B = (2 \times 2, \{\varnothing, \{(1,0),(1,1)\}, 2 \times 2\}, \{\varnothing, \{(0,1),(1,1)\}, 2 \times 2\})$.

Final remarks

- Motivated by the study of R. Noor *et al.*, which provided a characterization of Sierpinski object in the category of affine analogues of bitopological spaces of J. C. Kelly, we described Sierpinski object in the category of composite affine spaces.
- We have got a functor from the category of affine spaces to that of composite affine spaces, which preserves Sierpinski object.
- Instead of constructing Sierpinski object in the category of affine bitopological spaces from scratch, just apply our functor to the already available Sierpinski object for affine topological spaces.

Problem 29

Provide an analogous result for the case of composite affine systems.

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Thank you for your attention!