Uniform interpolation and coherence

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Equational deductive interpolation

Craig interpolation: if $\varphi \to \psi$, then there is a χ in the vocabulary common to φ and ψ, such that $\varphi \to \chi$ and $\chi \to \psi$.
A version for equations and varieties is below.

Definition

A variety $\mathcal{V}$ admits deductive interpolation if for any finite sets of variables $\overline{x}, \overline{y}, \overline{z}$ and any finite set of equations $\Sigma(\overline{x}, \overline{y}) \cup \{\varepsilon(\overline{y}, \overline{z})\}$ satisfying $\Sigma \models_{\mathcal{V}} \varepsilon$, there exists a finite set of equations $\Pi(\overline{y})$ such that $\Sigma \models_{\mathcal{V}} \Pi$ and $\Pi \models_{\mathcal{V}} \varepsilon$.

Equivalent formulation

$\mathcal{V}$ admits deductive interpolation if and only if for any finite sets $\overline{x}, \overline{y}$ and finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set (in general not finite) of equations $\Pi(\overline{y})$ such that for any equation $\varepsilon(\overline{y}, \overline{z})$:

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \Pi \models_{\mathcal{V}} \varepsilon.$$

Uniform deductive interpolation

Pitts uniform interpolation for intuitionistic logic: the interpolant χ only depends on the common vocabulary of φ and ψ.

A version for equations and varieties is below.

Definition

$\mathcal{V}$ admits (right) uniform deductive interpolation if for any finite sets of variables $\overline{x}, \overline{y}, \overline{z}$ and any finite set of equations $\Sigma(\overline{x}, \overline{y})$ there exists a finite set of equations $\Pi(\overline{y})$ such that $\Sigma \models_{\mathcal{V}} \Pi$ and for any equation $\varepsilon(\overline{y}, \overline{z})$:

$$\Sigma \models_{\mathcal{V}} \varepsilon \text{ implies } \Pi \models_{\mathcal{V}} \varepsilon.$$

Equivalent formulation

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Deductive interpolation and congruences

van Gool, Metcalfe and Tsinakis showed that $\mathcal{V}$ admits deductive interpolation if and only if the following diagram, involving lattices of congruences, commutes.

$$\begin{array}{ccc}
\mathrm{Con}\,\mathbf{F}(\overline{x},\overline{y}) & \xrightarrow{\;i^{-1}\;} & \mathrm{Con}\,\mathbf{F}(\overline{y}) \\
{\scriptstyle j^*}\big\downarrow & & \big\downarrow{\scriptstyle l^*} \\
\mathrm{Con}\,\mathbf{F}(\overline{x},\overline{y},\overline{z}) & \xrightarrow[\;k^{-1}\;]{} & \mathrm{Con}\,\mathbf{F}(\overline{y},\overline{z})
\end{array}$$

Here i, j, k, l are natural identity embeddings, j^* and l^* are corresponding natural extensions of compact congruences, and i^{-1}, k^{-1} are the corresponding natural restrictions of congruences.

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$$\begin{array}{ccc} \mathrm{KCon}\,\mathbf{F}(\overline{x},\overline{y}) & \xrightarrow{i^{-1}} & \mathrm{KCon}\,\mathbf{F}(\overline{y}) \\ \downarrow j^{*} & & \downarrow l^{*} \\ \mathrm{KCon}\,\mathbf{F}(\overline{x},\overline{y},\overline{z}) & \xrightarrow[k^{-1}]{} & \mathrm{KCon}\,\mathbf{F}(\overline{y},\overline{z}) \end{array}$$

Here i, j, k, l are natural identity embeddings, j^* and l^* are corresponding natural extensions of compact congruences, and i^{-1}, k^{-1} are the corresponding natural restrictions of compact congruences. But i^{-1}, k^{-1} are not always compact.

Connection to coherence

To make them always compact something slightly less than uniform deductive interpolation is needed.

Theorem (vGMT2017)

The following are equivalent:

1. *The compact lifting of any homomorphism between finitely presented algebras in $\mathcal{V}$ has a right adjoint.*
2. *For any finite sets $\overline{x}, \overline{y}$ and compact congruence Θ on $\mathbf{F}(\overline{x}, \overline{y})$, the congruence $\Theta \cap F(\overline{y})^2$ on $\mathbf{F}(\overline{y})$ is compact.*
3. *For any finite sets $\overline{x}, \overline{y}$ and finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Pi(\overline{y})$ such that for any equation $\varepsilon(\overline{y})$, we have $\Sigma \models_{\mathcal{V}} \varepsilon \iff \Pi \models_{\mathcal{V}} \varepsilon$.*

- (3) is implied by uniform deductive interpolation: $\varepsilon(\overline{y})$ is a special case of $\varepsilon(\overline{y}, \overline{z})$.

Connection to coherence

Condition (2) of the previous theorem

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Condition (2) of the previous theorem restated in terms of quotient algebras.

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- For any finitely presented algebra $\mathbf{B}$, certain finitely generated subalgebra of $\mathbf{B}$ is also finitely presented.

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A little more is true.

Theorem (Metcalfe, TK)

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2. *Finitely generated subalgebras of finitely presented algebras in $\mathcal{V}$ are themselves finitely presented.*

Coherence and model completeness

Definition

A class of structures $\mathcal{C}$ is coherent if for any $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ we have:

- if $\mathbf{B}$ is finitely presented, and $\mathbf{A} \leq \mathbf{B}$ is finitely generated, then $\mathbf{A}$ is finitely presented.

- Wheeler has shown that coherence is closely related to existence of a model companion of the first-order theory of $\mathcal{C}$.
- To be precise, Wheeler's results imply that the following are equivalent for a variety $\mathcal{V}$ with AP:
  - $\mathrm{Th}(\mathcal{V})$ has a model companion,
  - $\mathrm{Th}(\mathcal{V})$ has a model completion,
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  - $\mathrm{Th}(\mathcal{V})$ has a model completion,
  - $\mathcal{V}$ is coherent and $\mathcal{V}$ satisfies a weak form of CEP (conservative CEP for finite presentations).
- Very recently Metcalfe and Reggio characterised model completions for varieties of algebras in a much nicer way.

Examples and non-examples

- Every locally finite variety is coherent.
- The variety of Heyting algebras is coherent (follows from Pitt's interpolation thorem for intuitionistic propositional logic).
- The varieties of Abelian groups, lattice-ordered Abelian groups, and MV-algebras are coherent.
- The variety of all groups is not coherent (existence of a finitely generated recursively presented group that is not finitely presented + Higman's Theorem).
- For any variety with Higman Property, coherence is equivalent to:
  - Every finitely generated and recursively presented algebra is finitely presented. (Metcalfe, TK).

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- For any variety with Higman Property, coherence is equivalent to:
  - Every finitely generated and recursively presented algebra is finitely presented. (Metcalfe, TK).
- We found a criterion that produces a lot failures of coherence in ordered algebras.

A general criterion

Theorem

Let $\mathcal{V}$ be a coherent variety of algebras with a term-definable order, and a term $t(x,\overline{u})$ satisfying

$$\mathcal{V} \models t(x,\overline{u}) \leq x \quad \text{and} \quad \mathcal{V} \models x \leq y \;\Rightarrow\; t(x,\overline{u}) \leq t(y,\overline{u}).$$

Suppose also that $\mathcal{V}$ satisfies the following fixpoint embedding condition with respect to $t(x,\overline{u})$:

(FE) *For any finitely generated $\mathbf{A} \in \mathcal{V}$ and $a, \overline{b} \in A$, there exists $\mathbf{B} \in \mathcal{V}$ such that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\bigwedge_{k\in\mathbb{N}} t^k(a,\bar{b})$ exists in $\mathbf{B}$, satisfying*

$$\bigwedge_{k\in\mathbb{N}} t^k(a,\bar{b}) = t(\bigwedge_{k\in\mathbb{N}} t^k(a,\bar{b}),\bar{b}).$$

Then t is n-potent (with respect to x) in $\mathcal{V}$ for some $n \in \mathbb{N}$.

Applying the criterion

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- Satisfying (FE) may not be easy. It can be shown however that varieties closed under canonical extensions of their countable members, satisfy it. For example,
- Lattices are canonical. Consider the term

$$t(x, u, w) = (u \vee (w \wedge (u \vee x))) \wedge x.$$

For each $n \in \mathbb{N}$, there is a lattice $\mathbf{H}_n$ with $8 + 4n$ elements, which falsifies $t^n(x, u, w) \leq t^{n+1}(x, u, w)$.

Corollary (known since '80s)

Let $\mathcal{V}$ be a variety of lattices closed under canonical extensions and such that $\mathbf{H}_n \in \mathcal{V}$ for all $n \in \mathbb{N}$. Then $\mathcal{V}$ is not coherent.

More negative results

Theorem

Let $\mathcal{V}$ be a lattice-based variety of algebras, canonical, and having a term-operation which is not n-potent for any n. Then $\mathcal{V}$ is not coherent, does not admit uniform deductive interpolation and $Th(\mathcal{V})$ does not have model completion.

Examples:

- Varieties of residuated lattices. All "basic" ones: integral, commutative, involutive, square-increasing, distributive, semilinear, Hamiltonian, etc.
- Varieties of modal algebras. In particular any countably canonical variety whose Kripke frames admit all finite chains.
- Varieties of BAOs: countably canonical ones without EDPC.
- The variety of double Heyting algebras.

Thank you!

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The gadget

$$\Sigma = \{y \leq x, x \approx t(x, \overline{u}), x \leq z\} \quad \text{and} \quad \Pi = \{y \leq t^k(z, \overline{u}) \mid k \in \mathbb{N}\}$$

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First, we prove that for any equation $\varepsilon(y, z, \overline{u})$,

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- $\Sigma \models_{\mathcal{V}} \Pi$, so $\Leftarrow$ direction is immediate.
- Suppose the converse does not hold. Then, there is a finitely generated $\mathbf{A} \in \mathcal{V}$ and elements $a, b, \overline{c}$ of A such that $\mathbf{A} \models a \leq t^k(b, \overline{c})$ for every $k \in \mathbb{N}$, but $\mathbf{A} \not\models \varepsilon(a, b, \overline{c})$.

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- Now x does not occur in t or Π, so taking $x^{\mathbf{B}} = \bigwedge_{k\in\mathbb{N}} t^k(b,\bar{c})$ makes $\mathbf{B}$ a countermodel to $\Sigma \models_{\mathcal{V}} \varepsilon(y,z,\overline{u})$.

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- Now x does not occur in t or Π, so taking $x^{\mathbf{B}} = \bigwedge_{k \in \mathbb{N}} t^k(b, \bar{c})$ makes $\mathbf{B}$ a countermodel to $\Sigma \models_{\mathcal{V}} \varepsilon(y, z, \overline{u})$. Contradiction.

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- By coherence, there exists a finite $\Delta(y, z, \overline{u})$ such that $\Sigma \models_{\mathcal{V}} \varepsilon(y, z, \overline{u}) \iff \Delta \models_{\mathcal{V}} \varepsilon(y, z, \overline{u})$.

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- But we have $\Sigma \models_{\mathcal{V}} y \leq t^{m+1}(z, \overline{u})$.

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- Therefore, $\{y \leq t^m(z, \overline{u})\} \models_{\mathcal{V}} y \leq t^{m+1}(z, \overline{u})$.

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- And by monotonicity of t, we get $\mathcal{V} \models t^m(z, \overline{u}) \approx t^{m+1}(z, \overline{u})$.