The Minor Order for Homomorphisms via Natural Dualities

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Joint work with Bruno Teheux

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The Minor Order

Let $f\colon A^3 \to B$ be a function. Some minors of f:

- $g(x_1, x_2, x_3) = f(x_2, x_1, x_3)$
- $g(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3)$
- $g(x_1, x_2) = f(x_1, x_2, x_1)$

We write $g \preceq f$.

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- $g(x_1, x_2, x_3) = f(x_2, x_1, x_3)$ permuting arguments
- $g(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3)$ adding/deleting inessential arguments
- $g(x_1, x_2) = f(x_1, x_2, x_1)$ identifying arguments

We write $g \preceq f$.

$\Rightarrow$ preorder $\preceq$ on $\mathcal{F}_{AB} := \bigcup_{n\geq 1} Set(A^n, B)$

$\Rightarrow$ equivalence $\equiv$ on $\mathcal{F}_{AB}$

$\Rightarrow$ partial order $\leq$ on $\mathcal{F}_{AB}/_{\equiv}$

Restriction to Homomorphisms

We replace $\mathcal{F}_{AB}$ by $\mathcal{A}_{AB} := \bigcup_{n\geq 1} \mathcal{A}(A^n, B)$.

Here, $\mathcal{A}$ is a category of algebras with a natural duality:

$$\mathcal{A} \leftrightsquigarrow \text{ category of structured top. spaces } \mathcal{X}$$

$$A \in \mathcal{A} \leftrightsquigarrow A^* \in \mathcal{X}$$

$$f \in \mathcal{A}(A, B) \leftrightsquigarrow f^* \in \mathcal{X}(B^*, A^*)$$

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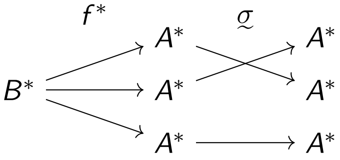
In particular, we assume that this duality is logarithmic:

$$A \times B \leftrightsquigarrow A^* \oplus B^*$$

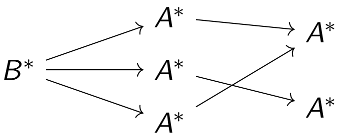
$$A^n \leftrightsquigarrow nA^*$$

Dual Minors

Let $f\colon A^3 \to B$ be a homomorphism with dual morphism $f^*\colon B^* \to 3A^*$.
Some dual minors of f^*:



permuting arguments

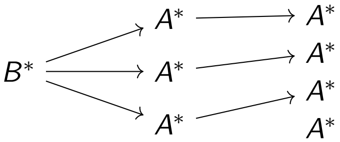


identifying arguments

Dual Minors

Proposition (Kerkhoff 2013)

The i-th argument of $f\colon A^n \to B$ is essential if and only if the dual morphism $f^*\colon B^* \to nA^*$ hits the i-th copy of A^*.



adding/deleting inessential arguments

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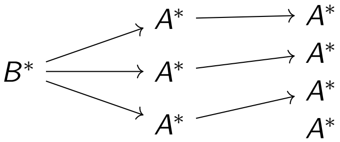

adding/deleting inessential arguments

- $\Rightarrow$ preorder $\preceq_d$ on $\mathcal{X}_{B^*A^*} := \bigcup_{n\geq 1} \mathcal{X}(B^*, nA^*)$
- $\Rightarrow$ $g \preceq f \Leftrightarrow g^* \preceq_d f^*$
- $\Rightarrow$ $\mathcal{A}_{AB}/_{\equiv}$ isomorphic to $\mathcal{X}_{B^*A^*}/_{\equiv_d}$

Finite Boolean Algebras

Let $2^j, 2^k$ be two finite Boolean algebras. We describe $\mathcal{BA}_{2^j,2^k}/_{\equiv}$.

Stone Duality:

$$2^k \leftrightsquigarrow \{1, \ldots, k\} =: [k]$$

$$f\colon (2^j)^n \to 2^k \leftrightsquigarrow f^*\colon [k] \to n[j]$$

Every map $[k] \to k[j]$ corresponds to some map $[k] \to [j]$, and modulo equivalence this is one-to-one.

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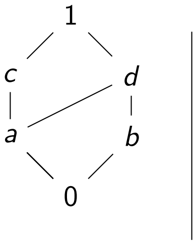
$$BA_{2^i,2^j}/_{\equiv} \simeq \biguplus_{[f^*] \text{ maximal}} [f^*]{\downarrow} \simeq \biguplus_{j^k} \Pi_k^{\partial}$$

The minor homomorphism poset is a disjoint union of (order-reversed) partition lattices.

Finite Distributive Lattices: Example

For $(L, \wedge, \vee)$, we want to determine $\mathcal{DL}_L/_{\equiv}$.

Priestley Duality yields $(L^*, \leq, 0, 1)$:



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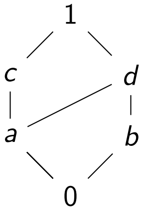
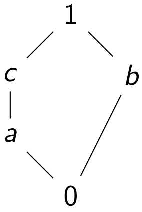


L



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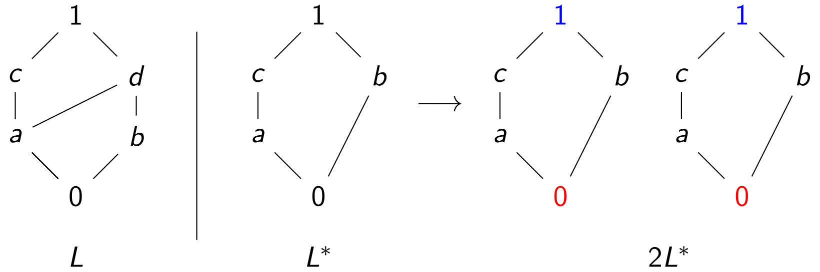

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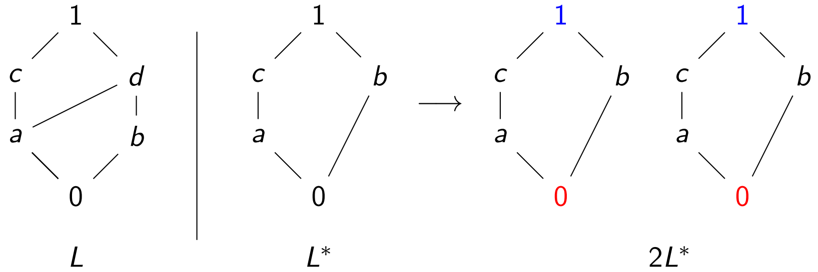

$$\mathcal{DL}_L/_\equiv \simeq \biguplus_{30} \Pi_2^\partial \uplus \biguplus_{29} \Pi_1^\partial \uplus \biguplus_{6} \Pi_0^\partial$$

Finite Distributive Latices: General Outline

Let $L \in \mathcal{DL}_{fin}$.

$$(L, \wedge, \vee) \leftrightsquigarrow (L^*, \leq, 0, 1)$$

- $\Rightarrow$ Look at the Hasse diagram of $L^* \backslash \{0, 1\}$
- $\Rightarrow$ Let $C_1, \ldots, C_n$ be its connected components
- $\Rightarrow$ $f^* \colon L^* \to L^*$ determines a maximal element $[F^*]$
- $\Rightarrow$ Essential arity of $F^* \colon L^* \to nL^*$ depends on $f^*(C_i)$

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$$\mathcal{DL}_L/_{\equiv} \simeq \biguplus_{f^* \colon L^* \to L^*} \Pi^{\partial}_{n - c_{f^*}}$$

where $c_{f^*} = \#\{i \mid f^*(C_i) \subseteq \{0, 1\}\}$

Weak Reconstructibility

A reconstruction problem:

$$f(x_1, x_2, x_3)$$

deck(f): $[f(x_1, x_1, x_2)]$ $[f(x_1, x_2, x_1)]$ $[f(x_1, x_2, x_2)]$

Weak Reconstructibility

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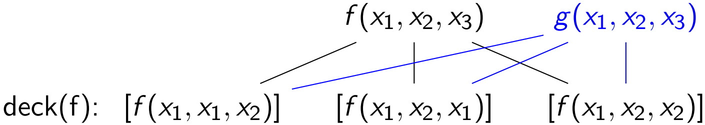

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- $\Rightarrow$ Let $f\colon A^n \to B$ be a homomorphism with $ess(f) > 2$
- $\Rightarrow$ Let $g\colon A^n \to B$ be a homomorphism with $deck(g) = deck(f)$
- $\Rightarrow$ Then $g \equiv f$.

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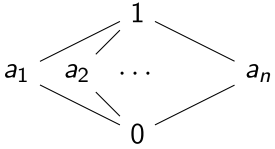
- $\Rightarrow$ Then $g \equiv f$.
- $\Rightarrow$ $\mathcal{A}_{AB}^{>2}$ is weakly reconstructible.

Finite Distributive Lattices: Boolean Reducts

Let $(L, \wedge, \vee) \in \mathcal{DL}_{\mathrm{fin}}$. Is L the reduct of a Boolean algebra?

Suppose $L = 2^n$:

$L^* =$



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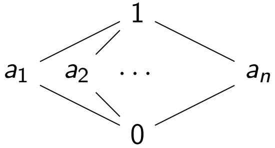

$$(nx + 2)^n = d_n x^n + \cdots + d_1 x + d_0$$

$$\mathcal{DL}_L/_{\equiv} \simeq \biguplus_{d_n} \Pi_n^{\partial} \uplus \cdots \uplus \biguplus_{d_1} \Pi_1^{\partial} \uplus \biguplus_{d_0} \Pi_0^{\partial}$$

Proposition

L is a Boolean reduct if and only if $\mathcal{DL}_L/_{\equiv}$ is of the above form for some $n \in \mathbb{N}$.

Infinite Algebras

In the infinite case topology comes into play.

Consider the Boolean algebra $B = \{X \subseteq \mathbb{N} \text{ finite or co-finite}\}$

$$B \leftrightsquigarrow \mathbb{N} \cup \{\infty\} \text{ one-point compactification}$$

$$f\colon B^n \to B \text{ homomorphism} \leftrightsquigarrow f^*\colon B^* \to nB^* \text{ continuous}$$

$\Rightarrow$ $\mathcal{BA}_B/_{\equiv}$ has uncountable antichains.

$\Rightarrow$ $\mathcal{BA}_B/_{\equiv}$ has countable chains.

Thanks for your attention!

W.Poiger, B.Teheux, *The Minor Order for Homomorphisms via Natural Duality* (2021)
https://arxiv.org/abs/2101.05545